On the size of shortest modal descriptions

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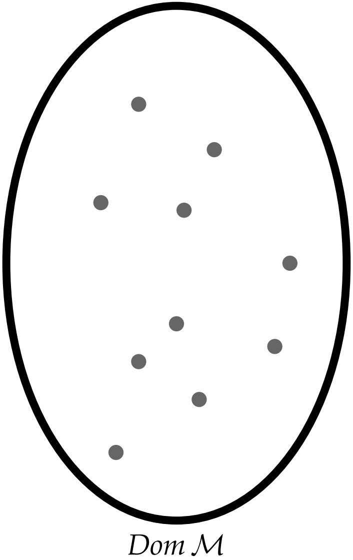
Separations and descriptions



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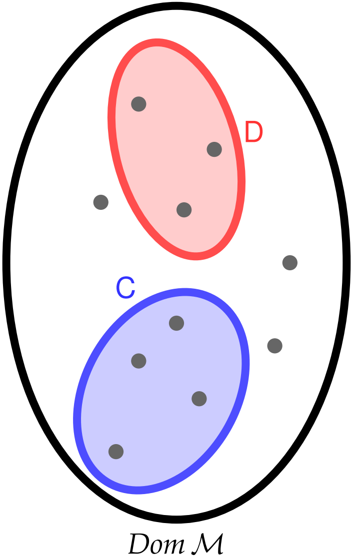

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- And let $\emptyset \neq \mathsf{C}, \mathsf{D} \subset \mathit{Dom}\,\mathcal{M}$

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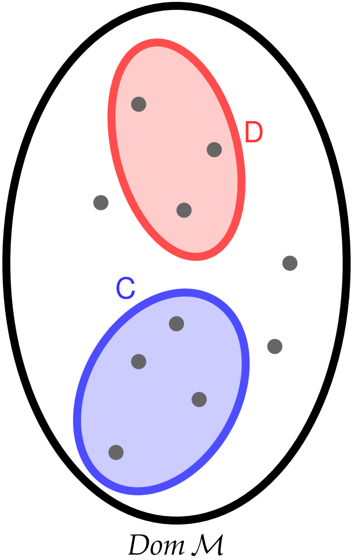

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- And let $\emptyset \neq \mathsf{C}, \mathsf{D} \subset \mathit{Dom}\,\mathcal{M}$
- For any formula φ, we say that:
  - φ *separates* C and D in $\mathcal{M}$ iff

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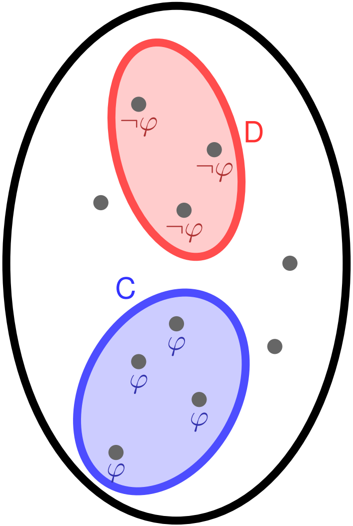

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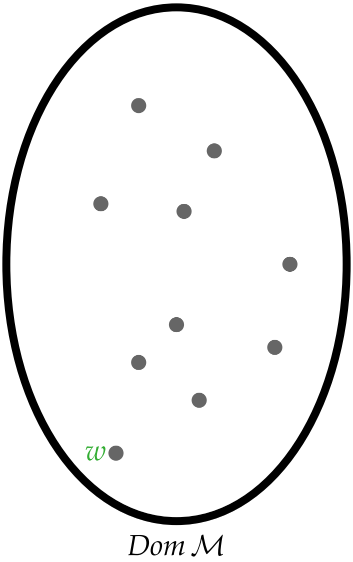

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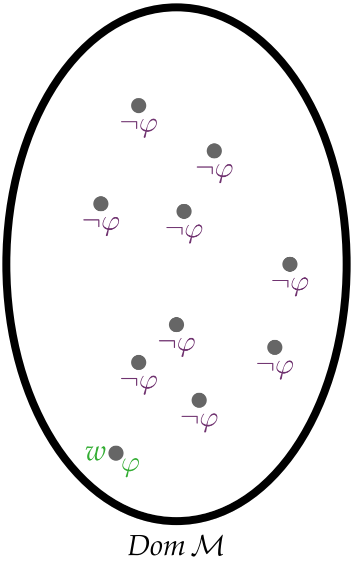

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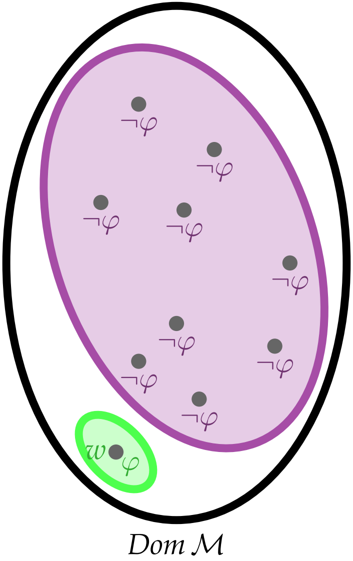

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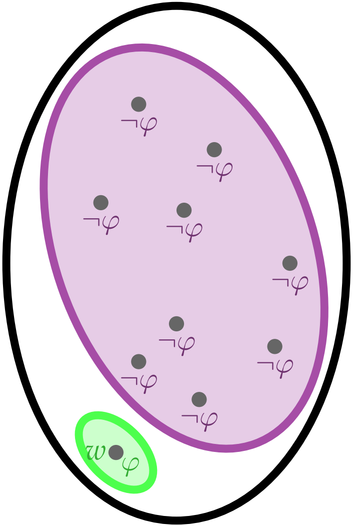

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- Description is a form of separation
- φ could be of any suitable logic

Separation and description problems

The separation problem

Given a finite model $\mathcal{M}$ and sets C, D $\subset$ *Dom* $\mathcal{M}$, find a φ that separates C and D, if possible.

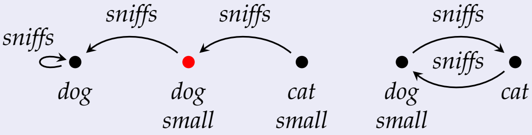
The description problem

Given a finite model $\mathcal{M}$ and a world $w \in$ *Dom* $\mathcal{M}$, find a φ that describes w, if possible.

- They can be seen as another kind of inference task
- But they didn't receive much attention so far
- We are interested in their computational properties

Motivation: Generation of Referring Expressions

An application of logics in Natural Language Generation



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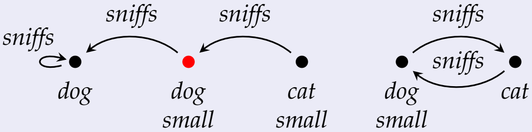

- "the dog that is sniffing another dog"

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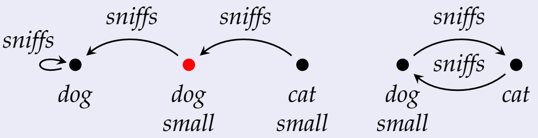

content determination

(e.g., as a FOL formula)

- $dog(x) \wedge \exists y.(x \not\approx y \wedge dog(y) \wedge sniffs(x, y))$

surface realization

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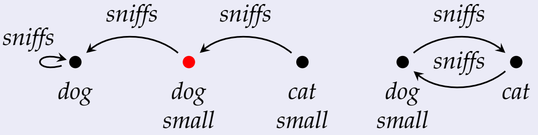

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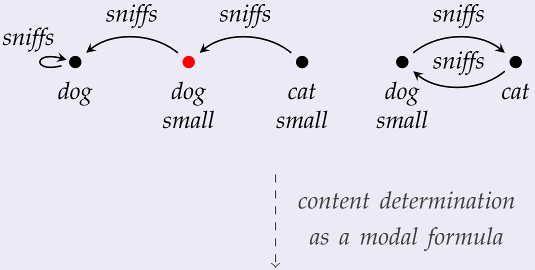

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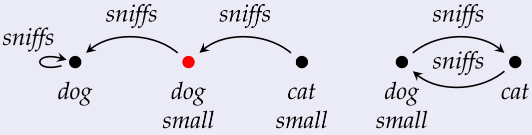


content determination as a modal formula

• $dog \wedge small \wedge \langle sniffs \rangle dog$

surface realization

• "the small dog that is sniffing a dog"

(logical) content determination ≈ description problem

Motivation: Modal logics in the GRE

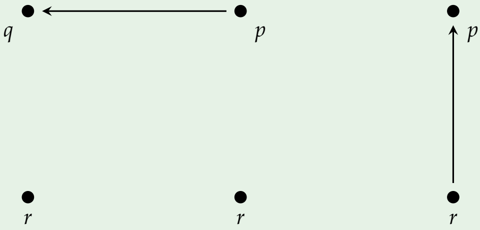
Areces, Koller & Striegnitz (2008)

- They propose modal logics for content determination:
  $\mathcal{ML}$ – the basic modal language $(\neg, \wedge, \Diamond)$
  $\mathcal{EL}$ – the existential positive fragment of $\mathcal{ML}$ $(\wedge, \Diamond)$
- *Rationale*:
  - Good expressive power
  - Simple surface realization algorithms
  - Relatively low computational complexity for inference tasks
- In particular, they show that:

  "The modal description problem needs polynomial time"

The modal description problem in polynomial time

A variation of Tarjan's bisimulation contraction algorithm



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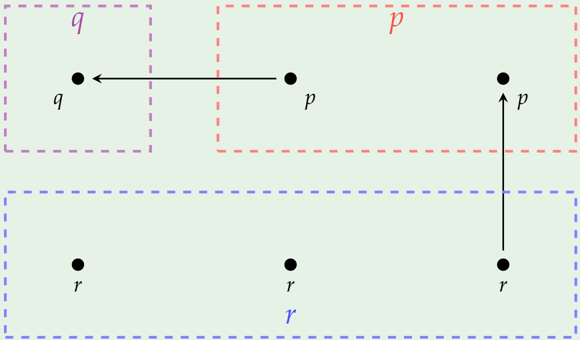

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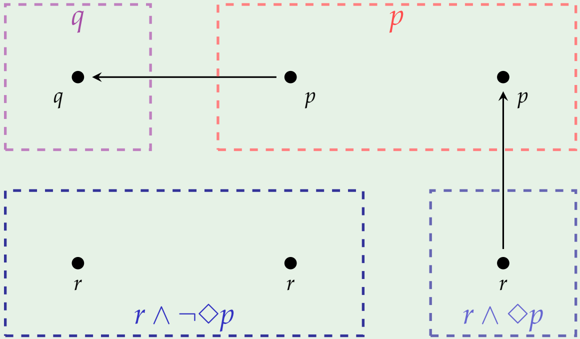

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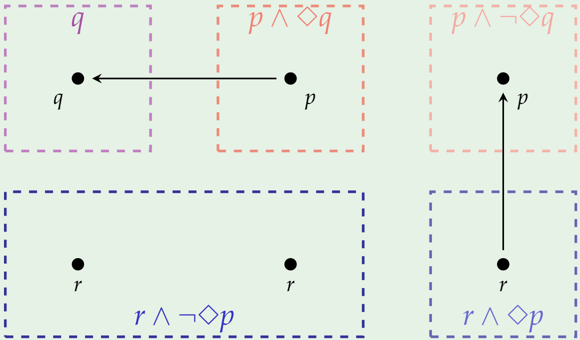

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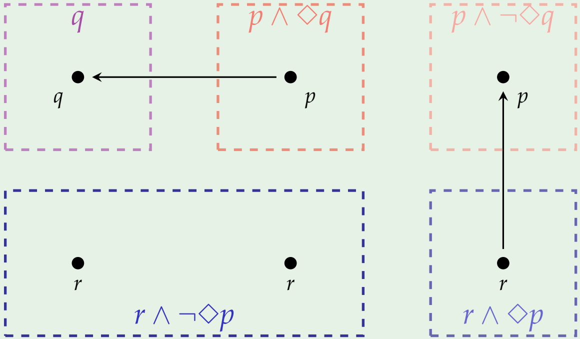

- Tarjan's algorithm runs in polynomial time
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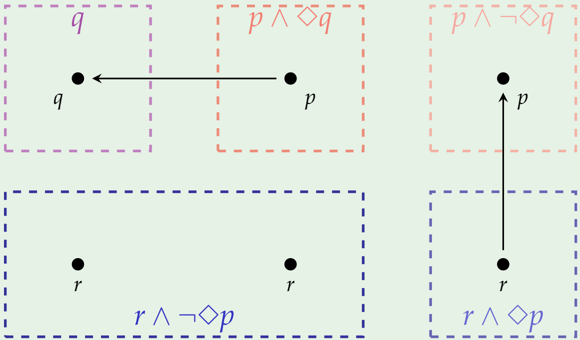

- Tarjan's algorithm runs in polynomial time
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- But this is assuming that $\wedge$ takes constant time!

The modal description problem in polynomial time

for DAG representation!

- This algorithm produces a formula represented as a DAG

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- Is the *tree* representation of this formula also polynomial?

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- The size of the DAG is polynomial in the size of the model
- Surface realization step doesn't exploit DAG representation
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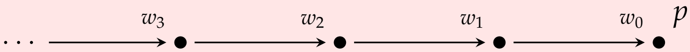
- Is the *tree* representation of this formula also polynomial?
- If not, "modal content determination" can't be said to take polynomial time

The modal description problem in polynomial time

also for tree representation?



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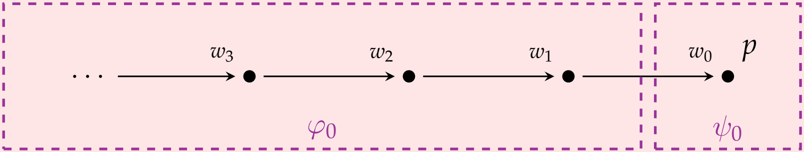

$\varphi_0 = \neg p$

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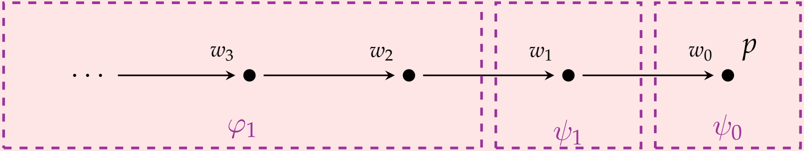

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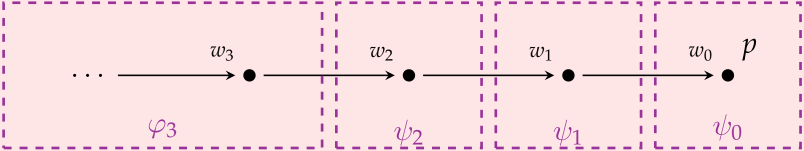

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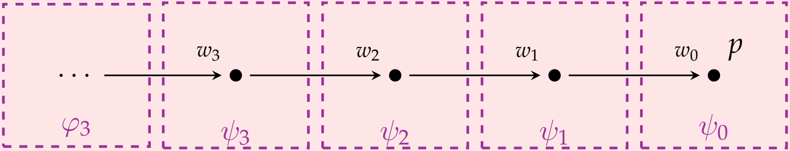

$$\begin{aligned} \varphi_0 &= \neg p \\ \varphi_1 &= \varphi_0 \wedge \neg\Diamond\psi_0 \\ \varphi_2 &= \varphi_1 \wedge \neg\Diamond\psi_1 \\ &\vdots \\ \varphi_{i+1} &= \varphi_i \wedge \neg\Diamond\psi_i \end{aligned}$$

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- Each ψ_i is description for w_i with size exponential in i

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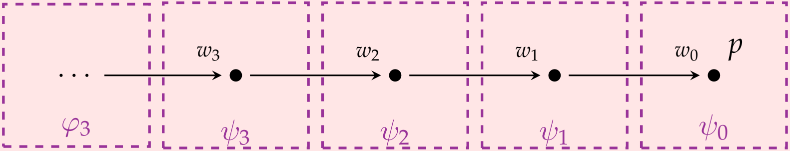

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&\vdots & &\vdots \\
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\end{aligned}
$$

- Each ψ_i is description for w_i with size exponential in i
- Observe that w_i admits a linear description: $\underbrace{\Diamond\Diamond\ldots\Diamond}_{i \text{ times}} p$

Where do we go from here?

- The example shows that this algorithm is not polynomial
- Can we *fix* it?
- Can we find *another* one that is indeed polynomial?

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- The example shows that this algorithm is not polynomial
- Can we *fix* it?
- Can we find *another* one that is indeed polynomial?
- We show that no such algorithm exists!

Bounds for the separation / description problems

Basic modal language $\mathcal{ML}$

Theorem (Lower bound)

Any upper bound for the size of a solution for the separation or description problem for $\mathcal{ML}$ is at least exponential.

Corollary

No polynomial time algorithm exists that solves the description or separation problem returning the formula as a tree.

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Theorem (Upper bound)

If $\varphi \in \mathcal{ML}$ is a minimum description for v in $\mathcal{M} = \langle W, R, V \rangle$, then $|\varphi| \in O(2^{\frac{1}{2}|W|^2} \cdot |V|)$.

Proof of the lower bound for $\mathcal{ML}$

Outline

Theorem (Lower bound)

Any upper bound for the size of a solution for the separation or description problem for $\mathcal{ML}$ is at least exponential.

Proof outline.

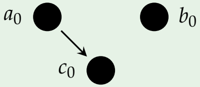
- Give a family of finite models with two distinguished points, $(\mathcal{M}_n, a_n, b_n)_{n \in \mathbb{N}}$, such that:
  i. $|\mathcal{M}_n| \in O(n)$
  ii. The shortest formula separating a_n and b_n is exponential in n
  iii. There exists a description of a_n in $\mathcal{M}_n$
- From i. and ii. the lower bound for separation follows
- This plus iii. proves the lower bound for description

□

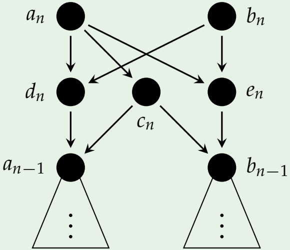
Proof of the lower bound for $\mathcal{ML}$

The (recursive) family of models

$(\mathcal{M}_0, a_0, b_0)$



$(\mathcal{M}_n, a_n, b_n), n > 0$



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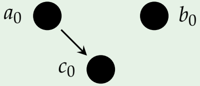

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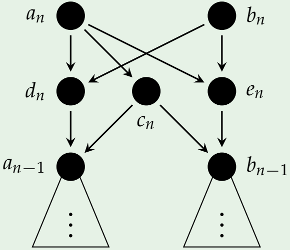

i. $|\mathcal{M}_n| = O(n)$

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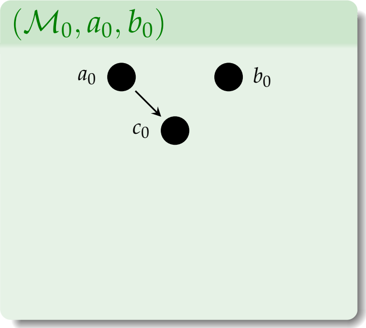

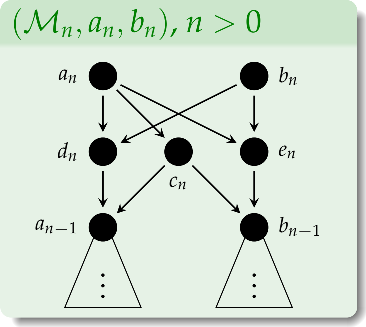

i. $|\mathcal{M}_n| = O(n)$

iii. If φ separates a_n and b_n, then $\varphi \wedge \Diamond^{2n+1}\top$ describes a_n in $\mathcal{M}_n$

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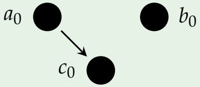

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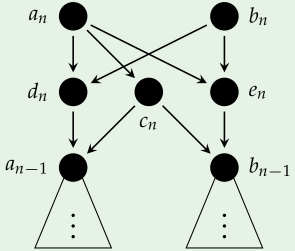

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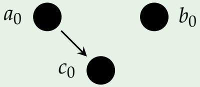

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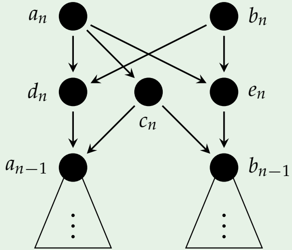

i. $|\mathcal{M}_n| = O(n)$

ii. We need to verify that

- a_n and b_n can be indeed separated by some ψ
- any such ψ will have size exponential in n

iii. If φ separates a_n and b_n, then $\varphi \wedge \Diamond^{2n+1}\top$ describes a_n in $\mathcal{M}_n$

Proof of the lower bound for $\mathcal{ML}$

Ehrenfeucht-Fraïssé games and strategies

- Ehrenfeucht-Fraïssé games: classical model-theoretical tool
- $\mathcal{G}_{\mathcal{M}}(a, b, n)$: n-turn game played by Spoiler and Duplicator
- Thm: Duplicator has a winning strategy for $\mathcal{G}_{\mathcal{M}}(a, b, n)$ iff no formula with n nested $\Diamond$'s separates a and b in $\mathcal{M}$

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- Dually, Spoiler has a winning strategy for $\mathcal{G}_{\mathcal{M}}(a, b, n)$ iff a formula with at most n nested $\Diamond$'s separates a and b in $\mathcal{M}$

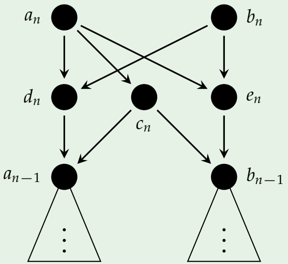
Proof of the lower bound for $\mathcal{ML}$

Ehrenfeucht-Fraïssé games and strategies

- Ehrenfeucht-Fraïssé games: classical model-theoretical tool
- $\mathcal{G}_{\mathcal{M}}(a, b, n)$: n-turn game played by Spoiler and Duplicator
- Thm: Duplicator has a winning strategy for $\mathcal{G}_{\mathcal{M}}(a, b, n)$ iff no formula with n nested $\Diamond$'s separates a and b in $\mathcal{M}$
- Dually, Spoiler has a winning strategy for $\mathcal{G}_{\mathcal{M}}(a, b, n)$ iff a formula with at most n nested $\Diamond$'s separates a and b in $\mathcal{M}$
- Moreover, we can relate the *size* of Spoiler's strategy with that of the separating formula

Proof of the lower bound for $\mathcal{ML}$

$(\mathcal{M}_n, a_n, b_n)$, $n > 0$



Spoiler's strategy for $\mathcal{G}(a_n, b_n, 2n+1)$

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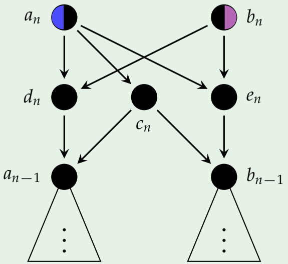

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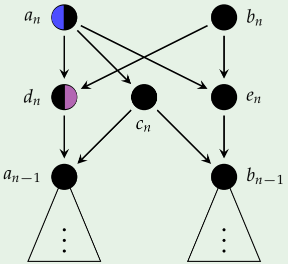

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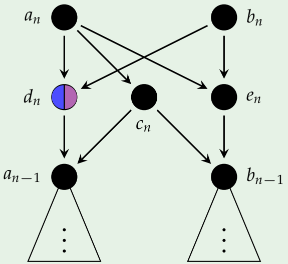

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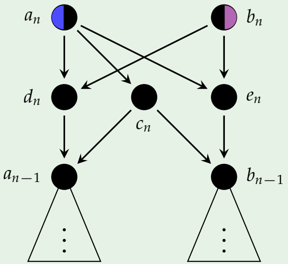

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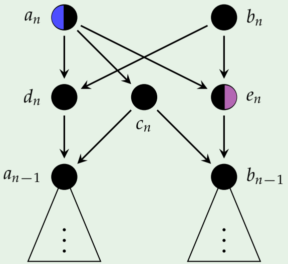

Spoiler's strategy for $\mathcal{G}(a_n, b_n, 2n + 1)$

Proof of the lower bound for $\mathcal{ML}$

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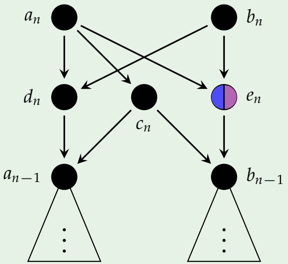

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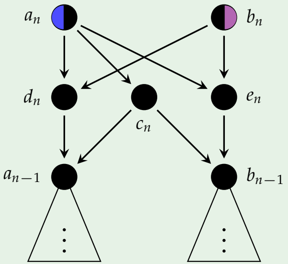

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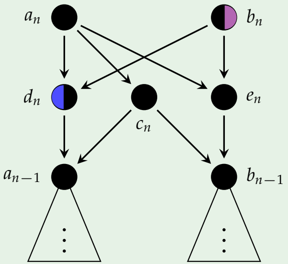

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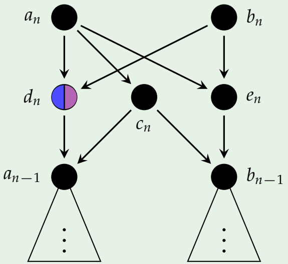

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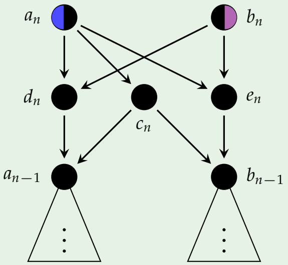

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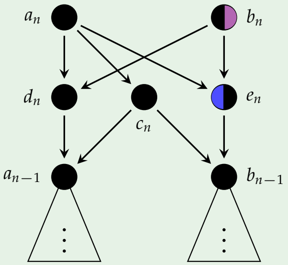

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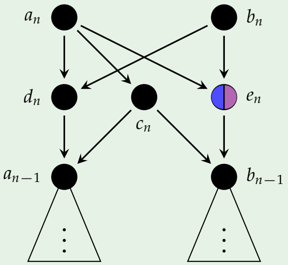

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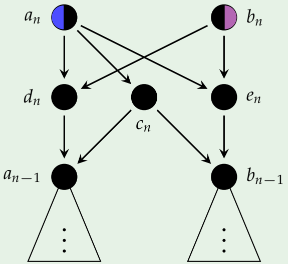

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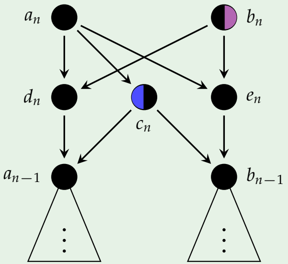
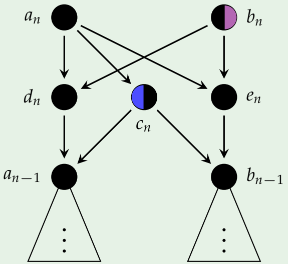


Spoiler's strategy for $\mathcal{G}(a_n, b_n, 2n+1)$

Pick left, play c_n

Proof of the lower bound for $\mathcal{ML}$

$(\mathcal{M}_n, a_n, b_n)$, $n > 0$



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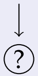

Proof of the lower bound for $\mathcal{ML}$

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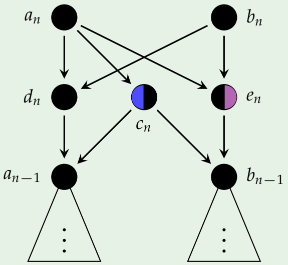

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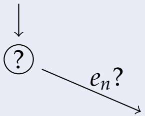

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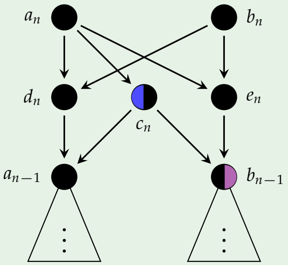

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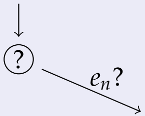

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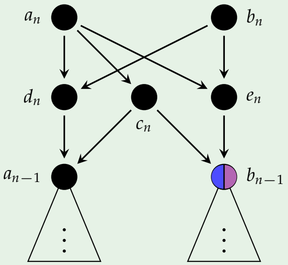

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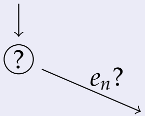

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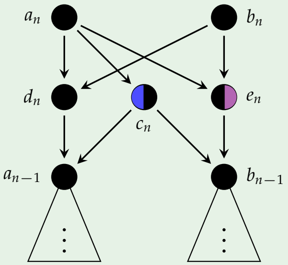

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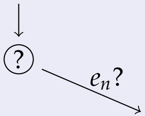

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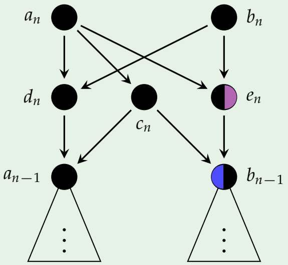

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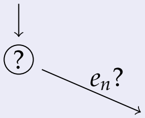

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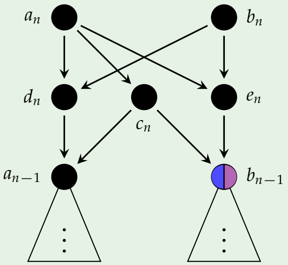

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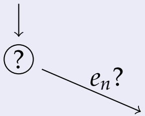

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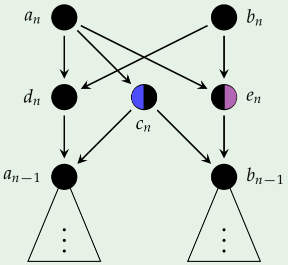

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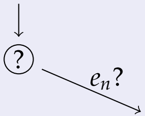

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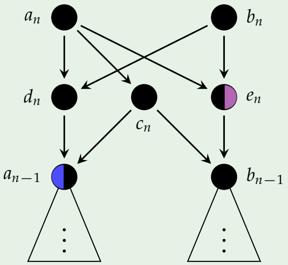


Spoiler's strategy for $\mathcal{G}(a_n, b_n, 2n+1)$

Pick left, play c_n

↓

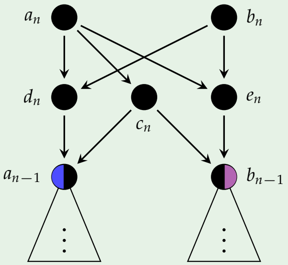
?

e_n?

Pick left, play a_{n-1}

Proof of the lower bound for $\mathcal{ML}$

$(\mathcal{M}_n, a_n, b_n)$, $n > 0$



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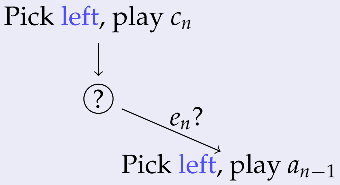

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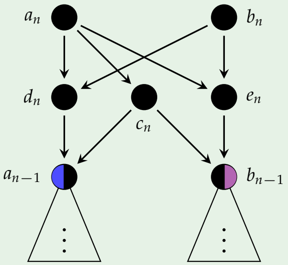


Spoiler's strategy for $\mathcal{G}(a_n, b_n, 2n+1)$

Pick left, play c_n

↓

?

e_n?

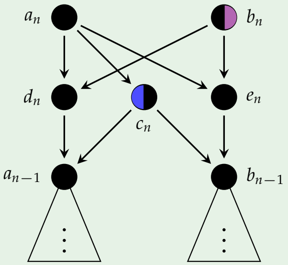
Pick left, play a_{n-1}

↓

Use strategy for $\mathcal{G}(a_{n-1}, b_{n-1}, 2n-1)$

Proof of the lower bound for $\mathcal{ML}$

$(\mathcal{M}_n, a_n, b_n)$, $n > 0$



Spoiler's strategy for $\mathcal{G}(a_n, b_n, 2n+1)$

Pick left, play c_n

↓

?

e_n?

Pick left, play a_{n-1}

↓

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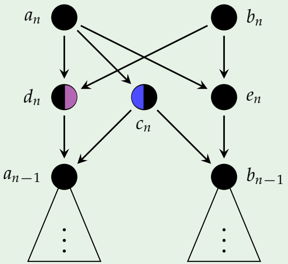

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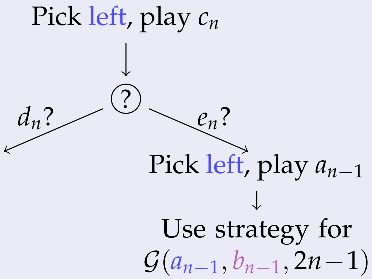

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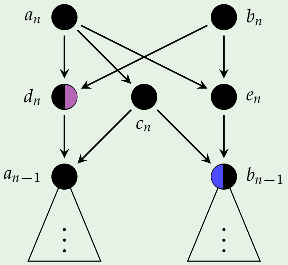

Spoiler's strategy for $\mathcal{G}(a_n, b_n, 2n+1)$

Pick left, play c_n

↓

?

d_n? → Pick left, play b_{n-1}

e_n? → Pick left, play a_{n-1}

↓

Use strategy for $\mathcal{G}(a_{n-1}, b_{n-1}, 2n-1)$

Proof of the lower bound for $\mathcal{ML}$

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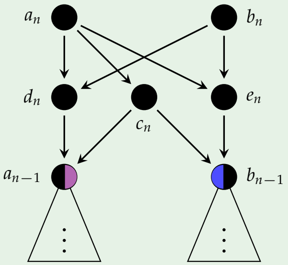

Spoiler's strategy for $\mathcal{G}(a_n, b_n, 2n+1)$

Pick left, play c_n

↓

?

d_n? → Pick left, play b_{n-1}

e_n? → Pick left, play a_{n-1}

↓

Use strategy for $\mathcal{G}(a_{n-1}, b_{n-1}, 2n-1)$

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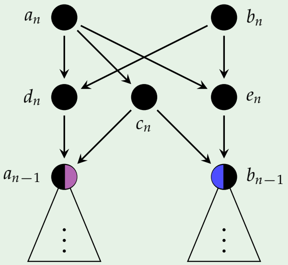
$(\mathcal{M}_n, a_n, b_n)$, $n > 0$



Spoiler's strategy for $\mathcal{G}(a_n, b_n, 2n+1)$

Pick left, play c_n

$\downarrow$

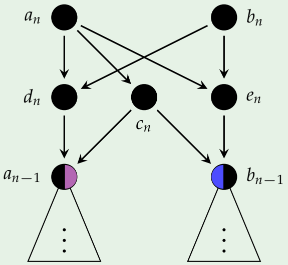
?

d_n? → Pick left, play b_{n-1} $\downarrow$ Use strategy for $\mathcal{G}(b_{n-1}, a_{n-1}, 2n-1)$

e_n? → Pick left, play a_{n-1} $\downarrow$ Use strategy for $\mathcal{G}(a_{n-1}, b_{n-1}, 2n-1)$

Proof of the lower bound for $\mathcal{ML}$

$(\mathcal{M}_n, a_n, b_n), n > 0$



Spoiler's strategy for $\mathcal{G}(b_n, a_n, 2n+1)$

Pick right, play c_n

↓

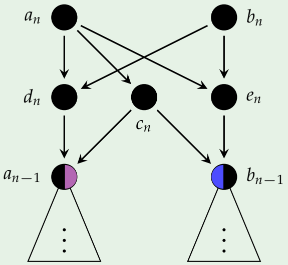
?

d_n? → Pick right, play b_{n-1} ↓ Use strategy for $\mathcal{G}(a_{n-1}, b_{n-1}, 2n-1)$

e_n? → Pick right, play a_{n-1} ↓ Use strategy for $\mathcal{G}(b_{n-1}, a_{n-1}, 2n-1)$

Proof of the lower bound for $\mathcal{ML}$

$(\mathcal{M}_n, a_n, b_n)$, $n > 0$



Spoiler's strategy for $\mathcal{G}(a_n, b_n, 2n+1)$

Pick left, play c_n

↓

?

d_n? → Pick left, play b_{n-1} ↓ Use strategy for $\mathcal{G}(b_{n-1}, a_{n-1}, 2n-1)$

e_n? → Pick left, play a_{n-1} ↓ Use strategy for $\mathcal{G}(a_{n-1}, b_{n-1}, 2n-1)$

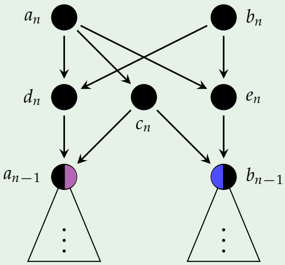
Intuitive argument

- The strategy induces φ_i, exponential in i, that separates a_i and b_i:

$$\varphi_0 = \Diamond\top \qquad \varphi_{n+1} = \Diamond(\Diamond\neg\varphi_n \wedge \Diamond\varphi_n)$$

Proof of the lower bound for $\mathcal{ML}$

$(\mathcal{M}_n, a_n, b_n)$, $n > 0$



Spoiler's strategy for $\mathcal{G}(a_n, b_n, 2n+1)$

Pick left, play c_n

↓

?

d_n? → Pick left, play b_{n-1} ↓ Use strategy for $\mathcal{G}(b_{n-1}, a_{n-1}, 2n-1)$

e_n? → Pick left, play a_{n-1} ↓ Use strategy for $\mathcal{G}(a_{n-1}, b_{n-1}, 2n-1)$

Intuitive argument

- The strategy induces φ_i, exponential in i, that separates a_i and b_i:

$$\varphi_0 = \Diamond\top \qquad \varphi_{n+1} = \Diamond(\Diamond\neg\varphi_n \wedge \Diamond\varphi_n)$$

- *A smaller separating formula would correspond to a smaller strategy.*
- But we saw that this is the only winning strategy for Spoiler!

Proof of the *upper* bound for $\mathcal{ML}$

Theorem (Upper bound)

If $\varphi \in \mathcal{ML}$ *is a minimum description for* v *in* $\mathcal{M} = \langle W, R, V \rangle$*, then* $|\varphi| \in O(2^{\frac{1}{2}|W|^2} \cdot |V|)$.

Proof idea.

Follows directly by analyzing the size of a formula generated by bisimulation contraction. □

What if we drop negation?

- AKS (2008) also investigate $\mathcal{EL}$ for content determination
  - Very weak fragment of $\mathcal{ML}$, only $\wedge$ and $\Diamond$ can be used
  - "Formulas with arbitrary negation are harder to realize"

What if we drop negation?

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- Could it be that $\mathcal{EL}$ has a polynomial description problem?
- We show it doesn't!

Bounds for the separation / description problems

$\mathcal{EL}$– the existential positive fragment of $\mathcal{ML}$

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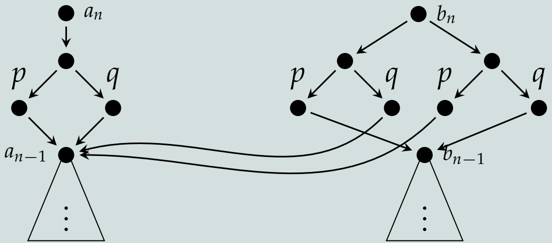
Proof idea.

Similar idea; the family of models is more complex, though:

$(\mathcal{N}_n, a_n, b_n), n > 0$



□

Succinctness

A quantitative way of comparing logics

“$\mathcal{L}_1$ is *more succinct* than $\mathcal{L}_2$”

- Fix a family of properties $(\Phi_n)_{n\in\mathbb{N}}$
- Express each Φ_n with a “short” formula φ_n of $\mathcal{L}_1$
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" $\mathcal{L}_1$ is *more succinct for separation* than $\mathcal{L}_2$"

- Fix a family of models and sets $(\mathcal{M}_n, C_n, D_n)_{n\in\mathbb{N}}$
- Separate C_n and D_n in $\mathcal{M}_n$ with a "short" formula φ_n of $\mathcal{L}_1$
- Prove: any ψ_n of $\mathcal{L}_2$ that separates C_n and D_n in $\mathcal{M}_n$ is "much larger" than φ_n
- φ_n and ψ_n need not be *logically equivalent*

Quantitative comparison of non-equivalent logics

Notation – $\mathcal{L}_1 \ll \mathcal{L}_2$

"$\mathcal{L}_1$ is exponentially more succinct for separation than $\mathcal{L}_2$"

Theorem

$$\mathcal{FO}^{=} \ll \mathcal{ML} \ll \mathcal{EL}^{\star} \ll \mathcal{EL}$$

$\mathcal{FO}^{=}$ First-order logic + equality (correspondence language)

$\mathcal{EL}^{\star}$ $\mathcal{EL}$ + atomic negation

Future work

- Computing the shortest formula that separate or describe
  - Application to Natural Language Generation
  - Good algorithms?
  - What's the complexity?
- Study other logics: relation with equality?
- Links with Kolmogorov complexity?

Aside: Колмогóров Complexity and Logics

What's the size of the shortest $\mathcal{L}$-formula that expresses property Φ?

$$E_{\mathcal{L}}(\Phi) = \min\{|\varphi| : \varphi \in \mathcal{L}, \mathcal{M} \text{ verifies } \Phi \text{ iff } \mathcal{M} \models \varphi\} \cup \{\infty\}$$

Kolmogorov Complexity: What's size of the the shortest M-program that computes string σ?

$$K_M(\sigma) = \min\{|p| : M(p) = \sigma\} \cup \{\infty\}$$

Algorithmic side

- Meaning of a program p:

  the output of p in M

- This meaning depends on M
- M extends M' then $K_M \leq K_{M'}$

Logical side

- Meaning of a formula φ:

  $\{\mathcal{M} : \mathcal{M} \models_{\mathcal{L}} \varphi\}$

- This meaning depends on $\mathcal{L}$
- $\mathcal{L}$ extends $\mathcal{L}'$ then $E_{\mathcal{L}} \leq E_{\mathcal{L}'}$

- Connections between Kolmogorov complexity and Logics?
- Transfer results from Kolmogorov complexity theory to Logics?
- How about other logical 'tasks' such as separation or description?