Complexity of the Lambek Calculus and Its Fragments

Mati Pentus
http://lpcs.math.msu.su/~pentus/

- The **Lambek calculus** (denoted L) is a mathematical tool for formal language specification. It generates the class of all context-free languages without the empty word.
- The **Lambek calculus with empty antecedents** (denoted L^*) generates the class of all context-free languages.
- **Proof nets** provide a convenient criterion for derivability in L^*.
- The derivability problems for $L^*(\backslash, /)$ and $L(\backslash, /)$ are **NP-complete**.
- The derivability problems for $L^*(\backslash)$ and $L(\backslash)$ are decidable in **deterministic polynomial time**.

1 "Hilbert style" Lambek calculus

J. Lambek, *The mathematics of sentence structure*,
American Mathematical Monthly **65** (1958), no. 3, 154–170.

Definition 1. The set of all *types* is defined as the minimal set Tp such that

- $\{p_0, p_1, p_2, \ldots\} \subset \mathrm{Tp}$
- If $A \in \mathrm{Tp}$ and $B \in \mathrm{Tp}$, then
  $(A \cdot B) \in \mathrm{Tp}$, $(A \backslash B) \in \mathrm{Tp}$, and $(A / B) \in \mathrm{Tp}$.

Example 1. $(p_1 \cdot (p_1 \backslash p_2)) \in \mathrm{Tp}$.

Below, we shall use q, r, s, and t instead of p_0, p_1, p_2, and p_3.

Derivable objects of $\mathrm{L_H}$ are $A \to B$, where $A \in \mathrm{Tp}$ and $B \in \mathrm{Tp}$.

Example 2. $\mathrm{L_H} \vdash (s/q) \cdot q \to s$, but $\mathrm{L_H} \nvdash q \cdot (s/q) \to s$.

Axioms and rules of $\mathrm{L_H}$ (the "Hilbert style" Lambek calculus)

$$A \to A \qquad (A\cdot B)\cdot C \to A\cdot(B\cdot C) \qquad A\cdot(B\cdot C) \to (A\cdot B)\cdot C$$

$$\frac{A \to B \quad B \to C}{A \to C} \qquad \frac{A\cdot B \to C}{A \to C/B} \qquad \frac{A\cdot B \to C}{B \to A\backslash C}$$

$$\frac{A \to C/B}{A\cdot B \to C} \qquad \frac{B \to A\backslash C}{A\cdot B \to C}$$

We write $\mathrm{L_H} \vdash A \to B$ for "$A \to B$ is derivable in the calculus $\mathrm{L_H}$".

Example 3. $\mathrm{L_H} \vdash q \to s/(q\backslash s)$.

$$\cfrac{\cfrac{q\backslash s \to q\backslash s}{q\cdot(q\backslash s) \to s}}{q \to s/(q\backslash s)}$$

Remark. $\mathrm{L_H} \nvdash s/(q\backslash s) \to q$.

Example 4.

$$\mathrm{L_H} \vdash A \cdot (A \backslash B) \to B, \qquad \mathrm{L_H} \vdash (B/A) \cdot A \to B,$$
$$\mathrm{L_H} \vdash (A \backslash B) \cdot (B \backslash C) \to A \backslash C, \qquad \mathrm{L_H} \vdash (C/B) \cdot (B/A) \to C/A,$$
$$\mathrm{L_H} \vdash A \to B/(A \backslash B), \qquad \mathrm{L_H} \vdash A \to (B/A) \backslash B,$$
$$\mathrm{L_H} \vdash (A \backslash B)/C \to A \backslash (B/C), \qquad \mathrm{L_H} \vdash A \backslash (B/C) \to (A \backslash B)/C.$$

Definition 2. $A \leftrightarrow B$ iff $\mathrm{L_H} \vdash A \to B$ and $\mathrm{L_H} \vdash B \to A$.

Example 5.

$$(A \backslash B)/C \leftrightarrow A \backslash (B/C),$$
$$A/(B \cdot C) \leftrightarrow (A/C)/B,$$
$$A \cdot (A \backslash (A \cdot B)) \leftrightarrow A \cdot B,$$
$$A \backslash (A \cdot (A \backslash B)) \leftrightarrow A \backslash B.$$

Example 6.

$$\mathrm{L_H} \vdash ((r/q) \backslash s) \backslash t \to (r \backslash s) \backslash (q \backslash t),$$
$$\mathrm{L_H} \nvdash ((q \backslash r) \backslash s) \backslash t \to s \backslash ((r \backslash q) \backslash t),$$
$$\mathrm{L_H} \vdash ((q \backslash r) \backslash (q \backslash q)) \backslash t \to (q \backslash q) \backslash ((r \backslash q) \backslash t).$$

2 Gentzen style Lambek calculus

Tp^* denotes the set of all finite sequences of types.
Tp^+ denotes the set of all non-empty finite sequences of types.
Derivable objects of the calculus L (the Gentzen style Lambek calculus) are *sequents* $\Gamma \to A$, where $A \in \mathrm{Tp}$ and $\Gamma \in \mathrm{Tp}^+$.

Axioms and rules of L

$$A \to A$$

$$\frac{\Phi \to B \qquad \Gamma\, B\, \Delta \to A}{\Gamma\, \Phi\, \Delta \to A}\ (\text{cut})$$

$$\frac{A\, \Pi \to B}{\Pi \to A \backslash B}\ (\to \backslash)\ (\Pi \text{ is non-empty})$$

$$\frac{\Phi \to A \qquad \Gamma\, B\, \Delta \to C}{\Gamma\, \Phi\, (A \backslash B)\, \Delta \to C}\ (\backslash \to)$$

$$\frac{\Pi\, A \to B}{\Pi \to B / A}\ (\to /)\ (\Pi \text{ is non-empty})$$

$$\frac{\Phi \to A \qquad \Gamma\, B\, \Delta \to C}{\Gamma\, (B/A)\, \Phi\, \Delta \to C}\ (/ \to)$$

$$\frac{\Gamma \to A \qquad \Delta \to B}{\Gamma\, \Delta \to A \cdot B}\ (\to \cdot)$$

$$\frac{\Gamma\, A\, B\, \Delta \to C}{\Gamma\, (A \cdot B)\, \Delta \to C}\ (\cdot \to)$$

Here $A, B, C \in \mathrm{Tp}$ and $\Gamma, \Delta, \Phi, \Pi \in \mathrm{Tp}^*$.

Example 7. $\mathrm{L} \vdash A \cdot (B/C) \to (A \cdot B)/C$

$$\dfrac{A \to A \qquad \dfrac{C \to C \qquad B \to B}{(B/C)\, C \to B}\,(/\to)}{\dfrac{A\,(B/C)\,C \to (A \cdot B)}{\dfrac{A\,(B/C) \to (A \cdot B)/C}{A \cdot (B/C) \to (A \cdot B)/C}\,(\cdot\to)}\,(\to /)}\,(\to \cdot)$$

Theorem 1 (J. Lambek, 1958)**.**
$\mathrm{L} \vdash A_1 \ldots A_n \to B$ *if and only if* $\mathrm{L_H} \vdash A_1 \cdot \ldots \cdot A_n \to B$.

Theorem 2 (cut-elimination, J. Lambek, 1958)**.**
A sequent is derivable in L *if and only if it is derivable in* L *without* (cut).

Corollary. *The derivability problem for* L *(and for* $\mathrm{L_H}$*) is decidable in nondeterministic polynomial time.*

Remark. $\mathrm{L} \nvdash (A \cdot B)/C \to A \cdot (B/C)$.

3 Grammars

The purpose of a Lambek categorial grammar is to provide an algorithm for distinguishing sentences from nonsentences in a fragment of a natural language.

Example 8.

Mary	np
John	np
smiles	$np\backslash s$
sees	$(np\backslash s)/np$
charmingly	$(np\backslash s)\backslash(np\backslash s)$

$$np = p_1 \quad s = p_2$$

$$\dfrac{np \to np \qquad \dfrac{np \to np \qquad s \to s}{np\,(np\backslash s) \to s}\,(\backslash \to)}{\underset{\text{Mary}}{np} \quad \underset{\text{sees}}{((np\backslash s)/np)} \quad \underset{\text{John}}{np} \quad \to s}\,(/ \to)$$

$$\dfrac{(np\backslash s) \to (np\backslash s) \qquad \dfrac{np \to np \qquad s \to s}{np\,(np\backslash s) \to s}\,(\backslash \to)}{\underset{\text{Mary}}{np} \quad \underset{\text{smiles}}{(np\backslash s)} \quad \underset{\text{charmingly}}{((np\backslash s)\backslash(np\backslash s))} \quad \to s}\,(\backslash \to)$$

4 Lambek calculus with empty antecedents

Derivable objects of the calculus L^* are *sequents* $\Gamma \to A$, where $A \in \mathrm{Tp}$ and $\Gamma \in \mathrm{Tp}^*$ (Tp^* denotes the set of all finite sequences of types).

Axioms and rules of L^*

$$A \to A \qquad \frac{\Phi \to B \qquad \Gamma\, B\, \Delta \to A}{\Gamma\, \Phi\, \Delta \to A}\ (\mathrm{cut})$$

$$\frac{A\,\Pi \to B}{\Pi \to A\backslash B}\ (\to \backslash) \qquad \frac{\Phi \to A \qquad \Gamma\, B\, \Delta \to C}{\Gamma\, \Phi\, (A\backslash B)\, \Delta \to C}\ (\backslash \to)$$

$$\frac{\Pi\, A \to B}{\Pi \to B/A}\ (\to /) \qquad \frac{\Phi \to A \qquad \Gamma\, B\, \Delta \to C}{\Gamma\, (B/A)\, \Phi\, \Delta \to C}\ (/ \to)$$

$$\frac{\Gamma \to A \qquad \Delta \to B}{\Gamma\, \Delta \to A \cdot B}\ (\to \cdot) \qquad \frac{\Gamma\, A\, B\, \Delta \to C}{\Gamma\, (A \cdot B)\, \Delta \to C}\ (\cdot \to)$$

Example 9.

$$\frac{\dfrac{B \to B}{\to B\backslash B}\ (\to \backslash) \qquad A \to A}{A/(B\backslash B) \to A}\ (/ \to)$$

Cut-elimination theorem. *We may drop* (cut) *from* L^*.

5 Interpretation in the free group

If a sequent is derivable in L^*, then in the free group its translation is equal to the unit. Here $A/B = A \cdot B^{-1}$ and $A\backslash B = A^{-1} \cdot B$.

Example 10. $\mathrm{L}^* \vdash q \to (s/q)\backslash s$.

$$q\backslash((s/q)\backslash s) = q^{-1} \cdot ((s \cdot q^{-1})^{-1} \cdot s) = q^{-1} \cdot ((q \cdot s^{-1}) \cdot s) = q^{-1} \cdot q \cdot s^{-1} \cdot s = 1$$

6 Cyclic linear logic

Noncommutative linear logic was suggested by J.-Y. Girard in 1987 and expounded by D. N. Yetter.
D. N. Yetter, *Quantales and noncommutative linear logic*, Journal of Symbolic Logic, **55** (1990), no. 1, pp. 41–64.

Definition 3. Let $\mathrm{At} \leftrightharpoons \{p_0, p_1, p_2, \ldots\} \cup \{\overline{p_0}, \overline{p_1}, \overline{p_2}, \ldots\}$. *Linear formulas* are the elements of the minimal set Fm such that

- $\mathrm{At} \subset \mathrm{Fm}$,
- if $A \in \mathrm{Fm}$ and $B \in \mathrm{Fm}$, then $(A \otimes B) \in \mathrm{Fm}$ and $(A \parr B) \in \mathrm{Fm}$.

$$(p_i)^\perp \leftrightharpoons \overline{p_i} \qquad (\overline{p_i})^\perp \leftrightharpoons p_i$$

$$(A \otimes B)^\perp \leftrightharpoons (B)^\perp \parr (A)^\perp \qquad (A \parr B)^\perp \leftrightharpoons (B)^\perp \otimes (A)^\perp$$

The mapping $A \mapsto \widehat{A}$ embeds L^* into cyclic linear logic.

$$\begin{aligned}
\widehat{p} &= p\\
\widehat{A/B} &= \widehat{A} \parr (\widehat{B})^{\perp}\\
\widehat{A\backslash B} &= (\widehat{A})^{\perp} \parr \widehat{B}\\
\widehat{A \cdot B} &= \widehat{A} \otimes \widehat{B}
\end{aligned}$$

Example 11. $\widehat{q/(s \cdot r)} = q \parr (\overline{r} \parr \overline{s})$, and $\widehat{(q/r)/s} = (q \parr \overline{r}) \parr \overline{s}$.

Derivable objects of cyclic linear logic are *sequents*

$$\rightarrow A_1 \ \ldots \ A_n,$$

where $A_i \in \mathrm{Fm}$.
The intended meaning of $\rightarrow A_1 \ \ldots \ A_n$ is $A_1 \parr \ldots \parr A_n$.
Axioms and rules of CMLL

$$\rightarrow \overline{p_i}\, p_i \qquad\qquad \rightarrow p_i\, \overline{p_i}$$

$$\frac{\rightarrow \Gamma\, A\, B\, \Delta}{\rightarrow \Gamma\, (A \parr B)\, \Delta} \qquad \frac{\rightarrow \Gamma\, A \qquad \rightarrow \Phi\, B\, \Delta}{\rightarrow \Phi\, \Gamma\, (A \otimes B)\, \Delta} \qquad \frac{\rightarrow \Gamma\, A\, \Pi \qquad \rightarrow B\, \Delta}{\rightarrow \Gamma\, (A \otimes B)\, \Delta\, \Pi}$$

Example 12. CMLL $\vdash\ \to (\overline{p} \otimes q)\,(\overline{q} \otimes r)\,(\overline{r} \parr p)$.

$$\dfrac{\dfrac{\dfrac{\to \overline{p}\,p \qquad \to q\,\overline{q}}{\to (\overline{p} \otimes q)\,\overline{q}\,p} \qquad \to r\,\overline{r}}{\to (\overline{p} \otimes q)\,(\overline{q} \otimes r)\,\overline{r}\,p}}{\to (\overline{p} \otimes q)\,(\overline{q} \otimes r)\,(\overline{r} \parr p)}$$

Example 13. CMLL $\vdash\ \to (\overline{r} \otimes r)\,(\overline{r} \otimes r)\,(\overline{r} \parr r)$

Remark. $\mathrm{L}^* \vdash A_1 \ldots A_n \to B$ if and only if CMLL $\vdash\ \to \widehat{A_n}^{\perp} \ldots \widehat{A_1}^{\perp}\,\widehat{B}$.

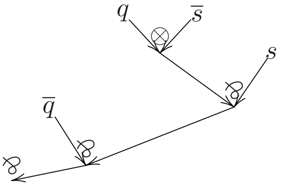
Example 14. $\mathrm{L}^* \vdash ((q\backslash r)\cdot s) \to (q\backslash(r\cdot s))$ and
CMLL $\vdash\ \to (\overline{s} \parr (\overline{r} \otimes q))\,(\overline{q} \parr (r \otimes s))$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\to \overline{r}\,r \qquad \to \overline{s}\,s}{\to \overline{s}\,\overline{r}\,(r \otimes s)} \qquad \to q\,\overline{q}}{\to \overline{s}\,(\overline{r} \otimes q)\,\overline{q}\,(r \otimes s)}}{\to \overline{s}\,(\overline{r} \otimes q)\,(\overline{q} \parr (r \otimes s))}}{\to (\overline{s} \parr (\overline{r} \otimes q))\,(\overline{q} \parr (r \otimes s))}$$

7 Region proof nets for CMLL and L^*

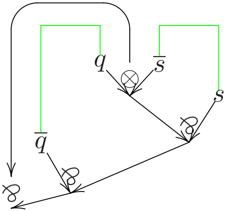
M. Pentus, *Free monoid completeness of the Lambek calculus allowing empty premises*, Proceedings of LC 1996, pp. 171–209.

For each sequent we build a tree. $q \to ((s/q)\backslash s) \quad \rightsquigarrow \quad q\backslash((s/q)\backslash s) \quad \rightsquigarrow$ $\overline{q} \parr \overline{s \parr \overline{q}} \parr s \quad \rightsquigarrow \quad \overline{q} \parr ((q \otimes \overline{s}) \parr s) \quad \rightsquigarrow$



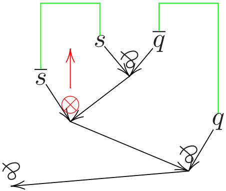
A region proof net consists of the tree, nonintersecting axiom links (green), and arcs leading from each $\otimes$ to a $\parr$ in the same region. The oriented graph consisting of black arcs must be acyclic.

Example 15. $\mathrm{L}^* \vdash q \to (s/q)\backslash s$.



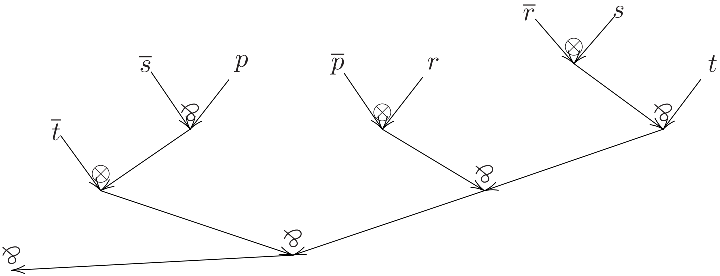
Example 16. $\mathrm{L}^* \nvdash (s/q)\backslash s \to q$.

$((s/q)\backslash s)\backslash q \quad \rightsquigarrow \quad \overline{\overline{s \parr \overline{q}}} \parr s \parr q \quad \rightsquigarrow \quad (\overline{s} \otimes (s \parr \overline{q})) \parr q$



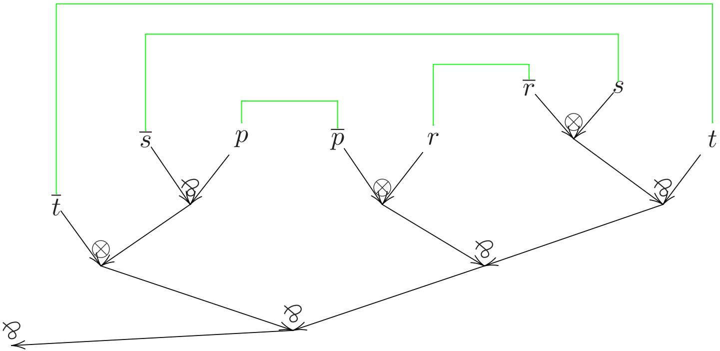
Example 17. $\mathrm{L}^* \vdash (s\backslash p)\backslash t \to (r\backslash p)\backslash((s\backslash r)\backslash t)$.

$\big((s\backslash p)\backslash t\big)\backslash\big((r\backslash p)\backslash((s\backslash r)\backslash t)\big) \quad \leadsto \quad \overline{\overline{\overline{s} \parr p} \parr t} \parr \overline{\overline{r} \parr p} \parr \overline{\overline{r} \parr p} \parr t \quad \leadsto$
$(\overline{t} \otimes (\overline{s} \parr p)) \parr (\overline{p} \otimes r) \parr (\overline{r} \otimes s) \parr t$

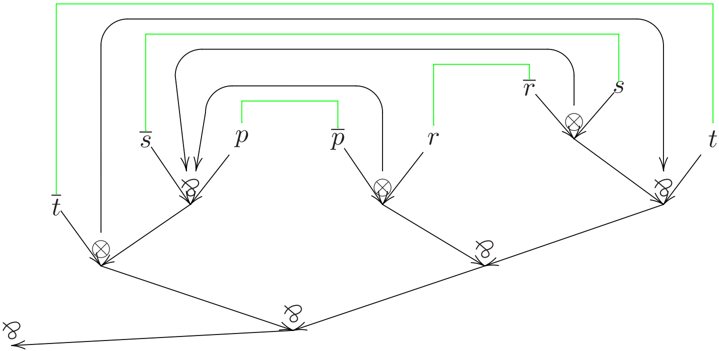


Example 17. $\mathrm{L}^* \vdash (s\backslash p)\backslash t \to (r\backslash p)\backslash((s\backslash r)\backslash t)$.

$\big((s\backslash p)\backslash t\big)\backslash\big((r\backslash p)\backslash((s\backslash r)\backslash t)\big) \quad\rightsquigarrow\quad \overline{\overline{\overline{s} \parr p} \parr t} \parr \overline{\overline{r} \parr p} \parr \overline{\overline{r} \parr p} \parr t \quad\rightsquigarrow$
$(\overline{t} \otimes (\overline{s} \parr p)) \parr (\overline{p} \otimes r) \parr (\overline{r} \otimes s) \parr t$



Example 17. $\mathrm{L}^* \vdash (s\backslash p)\backslash t \to (r\backslash p)\backslash((s\backslash r)\backslash t)$.
$\big((s\backslash p)\backslash t\big)\backslash\big((r\backslash p)\backslash((s\backslash r)\backslash t)\big) \quad \rightsquigarrow \quad \overline{\overline{\overline{s} \parr p} \parr t} \parr \overline{\overline{r} \parr p} \parr \overline{\overline{r} \parr p} \parr t \quad \rightsquigarrow$
$(\overline{t} \otimes (\overline{s} \parr p)) \parr (\overline{p} \otimes r) \parr (\overline{r} \otimes s) \parr t$



Theorem 3. *A sequent is derivable in* CMLL *if and only if there exists a region proof net for it.*

8 The complexity of L^* and L

M. Pentus, *Lambek calculus is* NP*-complete*, Theoretical Computer Science, **357** (2006), no. 1–3, pp. 186–201.

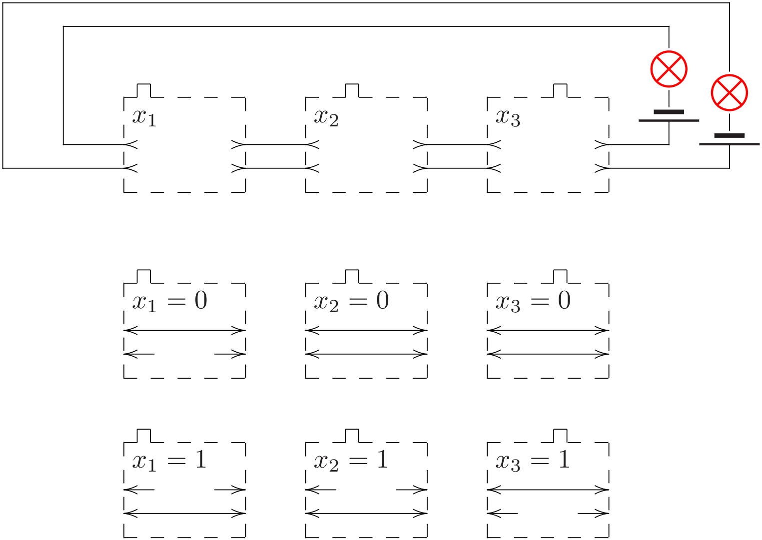
Remark. The derivability problem for CMLL is in NP.

Theorem 4 (2003)**.** *The derivability problems for* L^* *and* L *are* NP*-complete.*

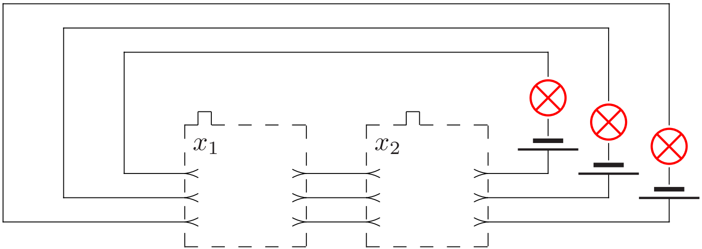
We can reformulate the well-known NP-complete problem SAT (satisfiability in the classical propositional logic) in terms of electrical circuits. This makes it easy to translate the problem into a problem concerning planar graphs.

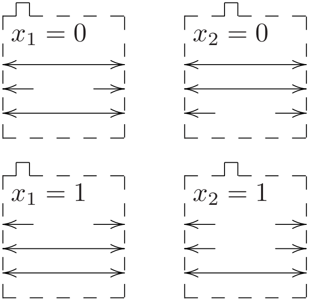
Let $c_1 \wedge \ldots \wedge c_m$ be a Boolean formula in conjunctive normal form with clauses $c_1, \ldots, c_m$ and variables $x_1, \ldots, x_n$.
We construct a frame (with m lamps and n sockets) and a set of $2n$ blocks (each of which fits into one socket only) so that the formula $c_1 \wedge \ldots \wedge c_m$ is satisfiable if and only if there is a way to plug n blocks into the sockets so that no lamp will be switched on. Each block (and each socket) has $2m$ contacts.

Example 18. $(x_1 \vee x_2) \wedge (\neg x_1 \vee x_3)$.



Example 19. $(x_1 \vee x_2) \wedge (\neg x_1 \vee x_2) \wedge (\neg x_2)$.





To model the circuits in L^* we shall construct types G, $E_i(0)$, $E_i(1)$, F_i (where $1 \leq i \leq n$) with the following properties.

- $\mathrm{L}^* \vdash E_1(t_1) \ldots E_n(t_n) \to G$ if and only if $\langle t_1, \ldots, t_n \rangle \in \{0,1\}^n$ is a satisfying assignment for the Boolean formula $c_1 \wedge \ldots \wedge c_m$.
- $\mathrm{L}^* \vdash F_1 \ldots F_n \to G$ if and only if $\mathrm{L}^* \vdash E_1(t_1) \ldots E_n(t_n) \to G$ for some $t_1, \ldots, t_n \in \{0,1\}$.

This can be done in deterministic polynomial time (in terms of the length of the given Boolean formula).
We construct these types from p_i^j (where $0 \leq i \leq n$ and $0 \leq j \leq m$).

For any Boolean variable x_i let $\neg_0 x_i$ stand for the literal $\neg x_i$ and $\neg_1 x_i$ stand for the literal x_i. Note that $\langle t_1, \ldots, t_n \rangle \in \{0,1\}^n$ is a satisfying assignment for the Boolean formula $c_1 \wedge \ldots \wedge c_m$ if and only if for every positive integer $j \leq m$ there exists a positive integer $i \leq n$ such that the literal $\neg_{t_i} x_i$ occurs in the clause c_j.

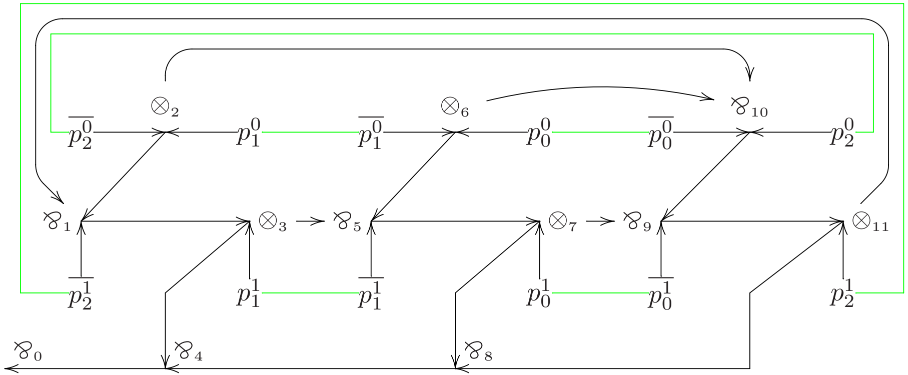
In the following definitions $1 \leq j \leq m$, $1 \leq i \leq n$ and $t \in \{0,1\}$.

$$
\begin{aligned}
G^0 &\leftrightharpoons p_0^0 \backslash p_n^0, \\
G^j &\leftrightharpoons (p_0^j \backslash G^{j-1}) \cdot p_n^j, \\
H_i^0 &\leftrightharpoons p_{i-1}^0 \backslash p_i^0, \\
H_i^j &\leftrightharpoons p_{i-1}^j \backslash (H_i^{j-1} \cdot p_i^j), \\
E_i^0(t) &\leftrightharpoons p_{i-1}^0 \backslash p_i^0, \\
E_i^j(t) &\leftrightharpoons \begin{cases} (p_{i-1}^j \backslash E_i^{j-1}(t)) \cdot p_i^j & \text{if } \neg_t x_i \text{ occurs in } c_j, \\ p_{i-1}^j \backslash (E_i^{j-1}(t) \cdot p_i^j) & \text{otherwise,} \end{cases} \\
G \leftrightharpoons G^m, \qquad & H_i \leftrightharpoons H_i^m, \qquad E_i(t) \leftrightharpoons E_i^m(t), \\
F_i &\leftrightharpoons (E_i(1)/H_i) \cdot H_i \cdot (H_i \backslash E_i(0)).
\end{aligned}
$$

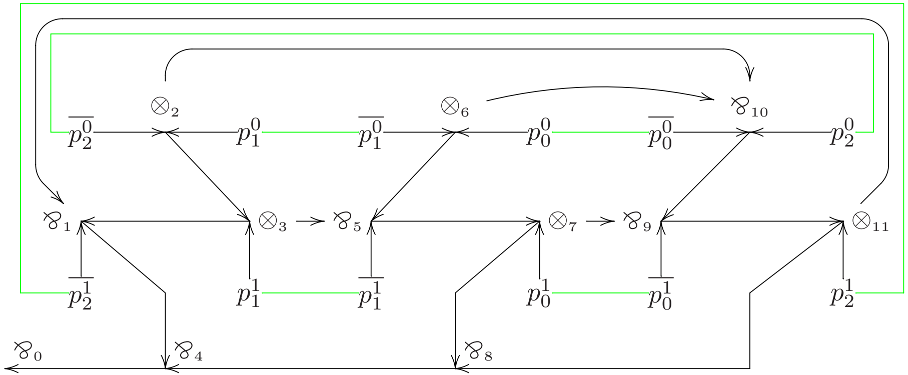
Example 20. Let $n = 2$, $m = 1$, and $c_1 = x_1 \vee x_2$.

$$
\begin{aligned}
&\langle 0,1 \rangle \qquad \mathrm{L}^* \vdash (p_0^1 \backslash ((p_0^0 \backslash p_1^0) \cdot p_1^1)) \; ((p_1^1 \backslash (p_1^0 \backslash p_2^0)) \cdot p_2^1) \to (p_0^1 \backslash (p_0^0 \backslash p_2^0)) \cdot p_2^1, \\
&\langle 0,0 \rangle \qquad \mathrm{L}^* \nvdash (p_0^1 \backslash ((p_0^0 \backslash p_1^0) \cdot p_1^1)) \; (p_1^1 \backslash ((p_1^0 \backslash p_2^0) \cdot p_2^1)) \to (p_0^1 \backslash (p_0^0 \backslash p_2^0)) \cdot p_2^1.
\end{aligned}
$$

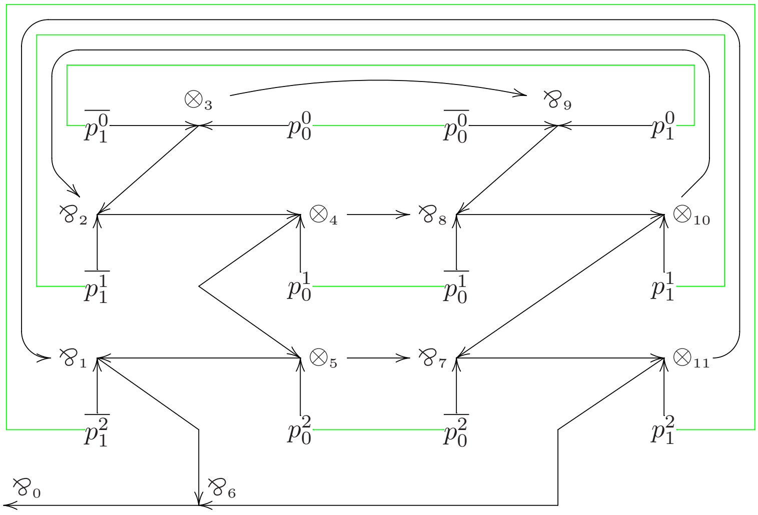
The Boolean formula $x_1 \vee x_2$ is true under the assignment $\langle 0,1 \rangle$ and false under the assignment $\langle 0,0 \rangle$.



The extended parse tree, axiom links, and region arcs corresponding to $x_1 \vee x_2$ and the truth assignment $\langle 0, 0 \rangle$.



The proof net corresponding to
$x_1 \vee x_2$ and the truth assignment $\langle 0, 1 \rangle$.



The extended parse tree, axiom links, and region arcs corresponding to $x_1 \wedge \neg x_1$ and the truth assignment $\langle 0 \rangle$.

9 The fragments with $\backslash$ and $/$

Theorem 5 (Y. Savateev, 2008). *The derivability problems for* $\mathrm{L}^*(\backslash, /)$ *and* $\mathrm{L}(\backslash, /)$ *are* NP*-complete.*

We shall construct types $\check{G}$, $\check{E}_i(0)$, $\check{E}_i(1)$ and sequences of types Γ_i with the following properties.

- $\mathrm{L}^*(\backslash, /) \vdash \check{E}_1(t_1) \ \ldots \ \check{E}_n(t_n) \to \check{G}$ if and only if $\langle t_1, \ldots, t_n \rangle \in \{0, 1\}^n$ is a satisfying assignment for the Boolean formula $c_1 \wedge \ldots \wedge c_m$.
- $\mathrm{L}^*(\backslash, /) \vdash \Gamma_1 \ \ldots \ \Gamma_n \to \check{G}$ if and only if $\mathrm{L}^* \vdash \check{E}_1(t_1) \ \ldots \ \check{E}_n(t_n) \to \check{G}$ for some $t_1, \ldots, t_n \in \{0, 1\}$.

We construct these types from p_i^j and q_i^j (where $0 \leq i \leq n$ and $0 \leq j \leq m$).

$$
\begin{aligned}
\check{G}^0 &\leftrightharpoons (p_0^0 \backslash p_n^0), \\
\check{G}^j &\leftrightharpoons (q_n^j / ((q_0^j \backslash p_0^j) \backslash \check{G}^{j-1})) \backslash p_n^j, \\
\check{C}_i^0 &\leftrightharpoons p_i^0, \\
\check{C}_i^j &\leftrightharpoons (q_i^j / \check{C}_i^{j-1}) \backslash p_i^j, \\
\check{E}_i^0(t) &\leftrightharpoons p_{i-1}^0, \\
\check{E}_i^j(t) &\leftrightharpoons \begin{cases} q_i^j / (((q_{i-1}^j / \check{E}_i^{j-1}(t)) \backslash p_{i-1}^j) \backslash p_i^{j-1}), & \text{if } \neg_t x_i \text{ occurs in } c_j, \\ (q_{i-1}^j / (q_i^j / (\check{E}_i^{j-1}(t) \backslash p_i^{j-1}))) \backslash p_{i-1}^j, & \text{otherwise,} \end{cases} \\
\check{F}_i(t) &\leftrightharpoons (\check{E}_i^m(t) \backslash p_i^m), \\
\Gamma_i &\leftrightharpoons (\check{F}_i(0) / (\check{C}_{i-1} \backslash \check{C}_i)) \, \check{F}_i(0) \, (\check{F}_i(0) \backslash \check{F}_i(1)), \\
\check{G} &\leftrightharpoons \check{G}^m.
\end{aligned}
$$

10 The fragments with $\cdot$ and $\backslash$

Theorem 6 (Y. Savateev, 2009)**.** *The derivability problems for* $\mathrm{L}^*(\cdot,\backslash)$, $\mathrm{L}(\cdot,\backslash)$, $\mathrm{L}^*(\cdot,/)$, *and* $\mathrm{L}(\cdot,/)$ *are* NP*-complete.*

$$G_i^0 \leftrightharpoons p_0^0 \backslash p_i^0,$$
$$G_i^j \leftrightharpoons (p_0^j \backslash G_i^{j-1}) \cdot p_i^j,$$
$$H_i^0 \leftrightharpoons p_{i-1}^0 \backslash p_i^0,$$
$$H_i^j \leftrightharpoons p_{i-1}^j \backslash (H_i^{j-1} \cdot p_i^j),$$
$$E_i^0(t) \leftrightharpoons p_{i-1}^0 \backslash p_i^0,$$
$$E_i^j(t) \leftrightharpoons \begin{cases} (p_{i-1}^j \backslash E_i^{j-1}(t)) \cdot p_i^j & \text{if } \neg_t x_i \text{ occurs in } c_j, \\ p_{i-1}^j \backslash (E_i^{j-1}(t) \cdot p_i^j) & \text{otherwise,} \end{cases}$$
$$H_i \leftrightharpoons H_i^m, \qquad E_i(t) \leftrightharpoons E_i^m(t),$$
$$\dot{F}_1 \leftrightharpoons E_1(0) \cdot ((E_1(0) \backslash E_1(1)) \cdot (H_1 \backslash r_1)),$$
$$\dot{F}_i \leftrightharpoons ((E_{i-1}(0) \backslash r_{i-1}) \backslash E_i(0)) \cdot (E_i(0) \backslash E_i(1)) \cdot (H_i \backslash r_i) \quad \text{if } 2 \le i \le n,$$
$$\dot{F}_{n+1} \leftrightharpoons (E_n(0) \backslash r_n) \backslash H_{n+1}.$$

We can prove that $\mathrm{L}^*(\cdot,\backslash) \vdash \dot{F}_1 \ldots \dot{F}_{n+1} \to G_{n+1}^m$ if and only if $\mathrm{L}^*(\cdot,\backslash) \vdash E_1(t_1) \ldots E_n(t_n) \to G_n^m$ for some $t_1, \ldots, t_n \in \{0,1\}$.

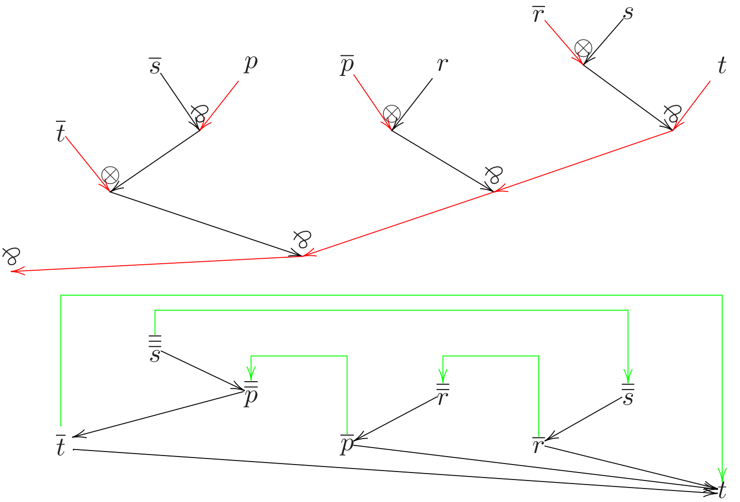
11 The fragments with one division

Theorem 7 (Y. Savateev, 2006, 2008)**.** *The derivability problems for* $\mathrm{L}^*(\backslash)$, $\mathrm{L}(\backslash)$, $\mathrm{L}^*(/)$, *and* $\mathrm{L}(/)$ *are decidable in deterministic polynomial time.*

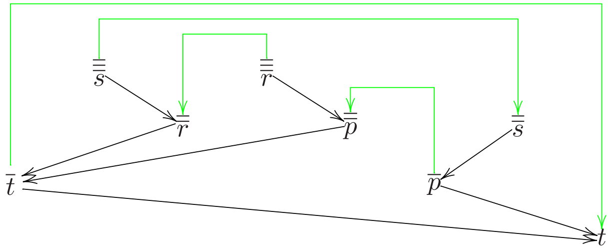
Y. Savateev, *The derivability problem for Lambek calculus with one division*, a talk at the International conference "Logical Models of Reasoning and Computation", May 2008.

Example 21. $\mathrm{L}^* \vdash (s\backslash p)\backslash t \to (r\backslash p)\backslash((s\backslash r)\backslash t)$.

$$\big((s\backslash p)\backslash t\big)\backslash\big((r\backslash p)\backslash((s\backslash r)\backslash t)\big) \quad \rightsquigarrow \quad \overline{\overline{\overline{s}\parr p}\parr t}\parr\overline{\overline{r}\parr p}\parr\overline{\overline{s}\parr r}\parr t$$



Example 22. $L^* \nvdash (r\backslash p)\backslash((s\backslash r)\backslash t) \to (s\backslash p)\backslash t$.



In the derivability criterion for $L^*(\backslash)$, we require the following.
(1) For each axiom link, the depth of its left end must be one greater than the depth of its right end.
(2) If the right end of an axiom link has depth $2k$, where $k > 0$, and the depth of the left predecessor of the left end is greater than $2k$, then there must be a vertex of depth less than $2k$ between the two ends of the axiom link.

In the derivability criterion for $L(\backslash)$, the second condition is made stronger by removing the red words.
(2') If the right end of an axiom link has non-zero even depth, then between the two ends of the axiom link there must be a vertex of smaller depth.

12 The fragments with one variable

$$A^k \leftrightharpoons \underbrace{A \cdot \ldots \cdot A}_{k \text{ times}}$$

Let a sequent contain only the variables $q_1, \ldots, q_{N-1}$. We replace q_i by $p^i \backslash p / p^{N-i}$, and this does not affect derivability.

Recall that $(A \cdot B) \backslash C \leftrightarrow B \backslash (A \backslash C)$ and $A/(B \cdot C) \leftrightarrow (A/C)/B$.

Example 23. Let $N = 3$. The substitution maps

$$q_1 \quad \text{to} \quad (p \backslash p/p)/p$$

and

$$q_2 \quad \text{to} \quad p \backslash (p \backslash p/p).$$

For example, we have $\mathrm{L} \nvdash q_1 \backslash q_2 \to q_1$ and

$$\mathrm{L} \nvdash ((p \backslash p/p)/p) \backslash (p \backslash (p \backslash p/p)) \to (p \backslash p/p)/p.$$

Let a sequent contain only the variables $q_1, \ldots, q_{N-1}$. We replace q_i by $\big(p^{i+1} \cdot \big(((p \cdot p)\backslash p)\backslash p\big) \cdot p^{N-i}\big)\backslash p$, and this does not affect derivability. (This substitution was discovered by S. Kuznetsov in 2007.)

Example 24. If $N = 3$, then Kuznetsov's substitution maps

$$q_1 \quad \text{to} \quad p\backslash(p\backslash(((p\backslash(p\backslash p))\backslash p)\backslash(p\backslash(p\backslash p))))$$

and

$$q_2 \quad \text{to} \quad p\backslash(((p\backslash(p\backslash p))\backslash p)\backslash(p\backslash(p\backslash(p\backslash p)))).$$

13 Conclusion

<table>
<tr><th></th><th>L</th><th>L^*</th><th>$L(p_1)$</th><th>$L^*(p_1)$</th></tr>
<tr><td>$\cdot, \backslash, /$</td><td>NP
2003</td><td>NP
2003</td><td>NP
2003</td><td>NP
2003</td></tr>
<tr><td>$\cdot, \backslash$</td><td rowspan="2">NP
2009</td><td rowspan="2">NP
2009</td><td rowspan="2">NP
2009</td><td rowspan="2">NP
2009</td></tr>
<tr><td>$\cdot, /$</td></tr>
<tr><td>$\cdot$</td><td>P</td><td>P</td><td>P</td><td>P</td></tr>
<tr><td>$\backslash, /$</td><td>NP
2008</td><td>NP
2008</td><td>NP
2008</td><td>NP
2008</td></tr>
<tr><td>$\backslash$</td><td rowspan="2">P
2006</td><td rowspan="2">P
2008</td><td rowspan="2">P
2006</td><td rowspan="2">P
2008</td></tr>
<tr><td>$/$</td></tr>
</table>