Proofs, disproofs, and their duals

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Abstract. Assertions, denials, proofs, disproofs, and their duals are discussed. Bi-intuitionistic logic, also known as Heyting-Brouwer logic, is extended in various ways by a strong negation connective that is used to express commitments arising from denials. These logics have been introduced and investigated in (Wansing 2008). In the present paper, a proof-theoretic semantics in terms of proofs, disproofs, and their duals is developed.

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I have a modest proposal: negation is denial *in the object language.*
Bryson Brown (2002)

	inferential status	related speech act
$\varnothing \vdash A$	A is provable direct verification	to assert that A
$\varnothing \vdash \sim A$	A is disprovable direct falsification	to deny that A
$A \vdash \varnothing$	A is reducible to non-truth indirect falsification	to assert that no information supports the truth of A
$\sim A \vdash \varnothing$	A is reducible to non-falsity indirect verification	to assert that no information supports the falsity of A

Table: Speech acts and the inferential status of propositions.

	inferential relation
$A_1, \ldots, A_n \vdash A$	A is provable from assumptions $A_1, \ldots, A_n$
$A_1, \ldots, A_n \vdash {\sim}A$	A is disprovable from assumptions $A_1, \ldots, A_n$
$A \vdash A_1, \ldots, A_n$	A is reducible to absurdity from counterassumptions $A_1, \ldots, A_n$
${\sim}A \vdash A_1, \ldots, A_n$	A is reducible to non-falsity from counterassumptions $A_1, \ldots, A_n$
${\sim}A_1, \ldots, {\sim}A_n \vdash A$	A is provable from rejections $A_1, \ldots, A_n$
${\sim}A_1, \ldots, {\sim}A_n \vdash {\sim}A$	A is disprovable from rejections $A_1, \ldots, A_n$
$A \vdash {\sim}A_1, \ldots, {\sim}A_n$	A is red. to absurdity from counterrejections $A_1, \ldots, A_n$
${\sim}A \vdash {\sim}A_1, \ldots, {\sim}A_n$	A is red. to non-falsity from counterrejections $A_1, \ldots, A_n$

Table: Inferential relations.

inferential relation	inferential status
$A_1, \ldots, A_n \vdash A$	$\varnothing \vdash (A_1 \wedge \ldots \wedge A_n) \rightarrow A$
$A_1, \ldots, A_n \vdash \sim A$	$\varnothing \vdash (A_1 \wedge \ldots \wedge A_n) \rightarrow \sim A$
$A \vdash A_1, \ldots, A_n$	$A \prec (A_1 \vee \ldots \vee A_n) \vdash \varnothing$
$\sim A \vdash A_1, \ldots, A_n$	$\sim A \prec (A_1 \vee \ldots \vee A_n) \vdash \varnothing$
$\sim A_1, \ldots, \sim A_n \vdash A$	$\varnothing \vdash (\sim A_1 \wedge \ldots \wedge \sim A_n) \rightarrow A$
$\sim A_1, \ldots, \sim A_n \vdash \sim A$	$\varnothing \vdash (\sim A_1 \wedge \ldots \wedge \sim A_n) \rightarrow \sim A$
$A \vdash \sim A_1, \ldots, \sim A_n$	$A \prec (\sim A_1 \vee \ldots \vee \sim A_n) \vdash \varnothing$
$\sim A \vdash \sim A_1, \ldots, \sim A_n$	$\sim A \prec (\sim A_1 \vee \ldots \vee \sim A_n) \vdash \varnothing$

Table: From inferential relations to inferential status.

A formula $C \prec B$ is to be read as "B co-implies C" or "C excludes B".

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Whereas implication is the residuum of conjunction, co-implication is the residuum of disjunction:

$$(A \wedge B) \vdash C \;\text{ iff }\; A \vdash (B \rightarrow C) \;\text{ iff }\; B \vdash (A \rightarrow C),$$

$$C \vdash (A \vee B) \;\text{ iff }\; (C \prec A) \vdash B \;\text{ iff }\; (C \prec B) \vdash A.$$

We arrive at the following vocabulary: $\{\wedge, \vee, \rightarrow, \prec, \sim\}$.

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Conjunction $\wedge$ combines formulas on the left of $\vdash$, and disjunction combines formulas on the right of $\vdash$. Implication is a vehicle for registering formulas that appear in antecedent position in succedent position, and co-implication is a vehicle for registering formulas that appear in succedent position in antecedent position.

The strong negation $\sim$ is a *primitive* negation. Other kinds of negation connectives are *definable* in the presence of $\rightarrow$ and $\prec$.

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Let p be a certain propositional letter. Then we define non-falsity as follows: $\top := (p \rightarrow p)$, and non-truth in this way: $\bot := (p \prec p)$. We can then introduce two negation connectives:

$$-A := (\top \prec A) \text{ (co-negation), and}$$

$$\neg A := (A \rightarrow \bot) \text{ (intuitionistic negation).}$$

Other defined connectives of HB are equivalence, $\leftrightarrow$, and co-equivalence, $\rightarrowtail\!\!\!\prec$, which are defined as follows:

$$A \equiv B := (A \rightarrow B) \wedge (B \rightarrow A);$$

$$A \rightarrowtail\!\!\!\prec B := (A \prec B) \vee (B \prec A).$$

Other defined connectives of HB are equivalence, $\leftrightarrow$, and co-equivalence, $\rightarrowtail\!\!\prec$, which are defined as follows:

$$A \equiv B := (A \to B) \wedge (B \to A);$$

$$A \rightarrowtail\!\!\prec B := (A \prec B) \vee (B \prec A).$$

The connectives $\wedge, \vee, \to$, and $\prec$ are the primitive connectives of bi-intuitionistic logic BiInt, alias Heyting-Brouwer logic HB. Extensions of HB by $\sim$ have been introduced and investigated in (Wansing 2008).

The propositional language $\mathcal{L}'$ of HB is defined in Backus–Naur form as follows:

$$\begin{aligned} \text{atomic formulas:} \quad & p \in Atom \\ \text{formulas:} \quad & A \in Form(Atom) \\ A ::= \quad & p \mid (A \wedge A) \mid (A \vee A) \mid (A \rightarrow A) \mid (A \prec A). \end{aligned}$$

It is well-known that intuitionistic propositional logic is faithfully embeddable into the modal logic S4 (= KT4), the logic of necessity and possibility on reflexive and transitive frames. The relational frame semantics of HB is simple and transparent. It reveals that HB can be faithfully embedded into temporal S4 (= $\mathsf{K}_t\mathsf{T4}$).

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Definition

A frame is a pre-order $\langle I, \leq \rangle$. Intuitively, I is a non-empty set of information states, and $\leq$ is a reflexive transitive binary relation of possible expansion of states on I.

Instead of $w \leq w'$, we also write $w' \geq w$.

Definition

An HB-model is a structure $\langle I, \leq, v^+ \rangle$, where $\langle I, \leq \rangle$ is a frame and v^+ is a function that maps every $p \in Atom$ to a subset of I. It is assumed that v^+ satisfies the following persistence (or heredity) condition for atoms:
if $w \leq w',$ then $w \in v^+(p)$ implies $w' \in v^+(p)$.

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if $w \leq w',$ then $w \in v^+(p)$ implies $w' \in v^+(p)$.

The relation $\mathcal{M}, w \models^+ A$ ('state w supports the truth of $\mathcal{L}'$-formula A in model $\mathcal{M}$') is inductively defined as follows:

$\mathcal{M}, w \models^+ p$	iff	$w \in v^+(p)$
$\mathcal{M}, w \models^+ (A \wedge B)$	iff	$\mathcal{M}, w \models^+ A$ and $\mathcal{M}, w \models^+ B$
$\mathcal{M}, w \models^+ (A \vee B)$	iff	$\mathcal{M}, w \models^+ A$ or $\mathcal{M}, w \models^+ B$
$\mathcal{M}, w \models^+ (A \rightarrow B)$	iff	for every $w' \geq w : \mathcal{M}, w' \not\models^+ A$ or $\mathcal{M}, w' \models^+ B$
$\mathcal{M}, w \models^+ (A \prec B)$	iff	there exists $w' \leq w : \mathcal{M}, w' \models^+ A$ and $\mathcal{M}, w' \not\models^+ B$

$$\mathcal{M}, w \models^{+} \neg A \quad \text{iff} \quad \text{for every } w' \geq w,\ \mathcal{M}, w' \not\models^{+} A;$$

$$\mathcal{M}, w \models^{+} -A \quad \text{iff} \quad \text{there exists } w' \leq w \text{ and } \mathcal{M}, w' \not\models^{+} A.$$

$$\mathcal{M}, w \models^+ \neg A \quad \text{iff} \quad \text{for every } w' \geq w,\ \mathcal{M}, w' \not\models^+ A;$$
$$\mathcal{M}, w \models^+ -A \quad \text{iff} \quad \text{there exists } w' \leq w \text{ and } \mathcal{M}, w' \not\models^+ A.$$

Observation (Persistence)

For every $\mathcal{L}'$-formula A, HB*-model $\langle I, \leq, v^+ \rangle$, and w, $w' \in I$:*

$$\textit{if } w \leq w',\ \textit{then } \mathcal{M}, w \models^+ A \textit{ implies } \mathcal{M}, w' \models^+ A.$$

$$\mathcal{M}, w \models^{+} \neg A \quad \text{iff} \quad \text{for every } w' \geq w,\ \mathcal{M}, w' \not\models^{+} A;$$

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$$\textit{if } w \leq w',\ \textit{then } \mathcal{M}, w \models^{+} A \textit{ implies } \mathcal{M}, w' \models^{+} A.$$

Definition

HB is the set of all $\mathcal{L}'$-formulas A such that for every HB-model $\langle I, \leq, v^{+}\rangle$, and $w \in I$: $\mathcal{M}, w \models^{+} A$.

The propositional language $\mathcal{L}$ is defined in Backus–Naur form as follows:

$$\begin{aligned} \text{atomic formulas:} &\quad p \in \mathit{Atom} \\ \text{formulas:} &\quad A \in \mathit{Form(Atom)} \\ A ::= &\quad p \mid {\sim}A \mid (A \wedge A) \mid (A \vee A) \mid (A \to A) \mid (A \prec A). \end{aligned}$$

Definition

A model is a structure $\langle I, \leq, v^+, v^- \rangle$, where $\langle I, \leq \rangle$ is a frame. Moreover, v^+ and v^- are functions that map every $p \in Atom$ to a subset of I (namely the states that support the truth of p and the falsity of p, respectively. The functions v^+ and v^- satisfy the following persistence conditions for atoms:
if $w \leq w',$ then $w \in v^+(p)$ implies $w' \in v^+(p)$;
if $w \leq w',$ then $w \in v^-(p)$ implies $w' \in v^-(p)$.

Definition (continued)

The relations $\mathcal{M}, w \models^+ A$ ('state w supports the truth of $\mathcal{L}$-formula A in model $\mathcal{M}$') and $\mathcal{M}, w \models^- A$ ('state w supports the falsity of $\mathcal{L}$-formula A in model $\mathcal{M}$') are inductively defined as follows:

$\mathcal{M}, w \models^+ p$	iff	$w \in v^+(p)$
$\mathcal{M}, w \models^- p$	iff	$w \in v^-(p)$
$\mathcal{M}, w \models^+ {\sim}A$	iff	$\mathcal{M}, w \models^- A$
$\mathcal{M}, w \models^- {\sim}A$	iff	$\mathcal{M}, w \models^+ A$
$\mathcal{M}, w \models^+ (A \wedge B)$	iff	$\mathcal{M}, w \models^+ A$ and $\mathcal{M}, w \models^+ B$
$\mathcal{M}, w \models^- (A \wedge B)$	iff	$\mathcal{M}, w \models^- A$ or $\mathcal{M}, w \models^- B$
$\mathcal{M}, w \models^+ (A \vee B)$	iff	$\mathcal{M}, w \models^+ A$ or $\mathcal{M}, w \models^+ B$
$\mathcal{M}, w \models^- (A \vee B)$	iff	$\mathcal{M}, w \models^- A$ and $\mathcal{M}, w \models^- B$
$\mathcal{M}, w \models^+ (A \rightarrow B)$	iff	for every $w' \geq w : \mathcal{M}, w' \not\models^+ A$ or $\mathcal{M}, w' \models^+ B$
$\mathcal{M}, w \models^+ (A \prec B)$	iff	there exists $w' \leq w : \mathcal{M}, w' \models^+ A$ and $\mathcal{M}, w' \not\models^+ B$.

In the following table, a number of support of falsity conditions for implications and co-implications are listed. For each choice of pairs of conditions, support of falsity is persistent for arbitrary formulas.

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cI_1	$\mathcal{M}, w \models^- (A \to B)$	iff $\mathcal{M}, w \models^+ A$ and $\mathcal{M}, w \models^- B$
cI_2	$\mathcal{M}, w \models^- (A \to B)$	iff for every $w' \geq w : \mathcal{M}, w' \not\models^+ A$ or $\mathcal{M}, w' \models^- B$
cI_3	$\mathcal{M}, w \models^- (A \to B)$	iff there is $w' \leq w : \mathcal{M}, w' \models^+ A$ and $\mathcal{M}, w' \not\models^+ B$
cI_4	$\mathcal{M}, w \models^- (A \to B)$	iff there is $w' \leq w : \mathcal{M}, w' \not\models^- A$ and $\mathcal{M}, w' \models^- B$
cC_1	$\mathcal{M}, w \models^- (A \prec B)$	iff $\mathcal{M}, w \models^- A$ or $\mathcal{M}, w \models^+ B$
cC_2	$\mathcal{M}, w \models^- (A \prec B)$	iff there is $w' \leq w : \mathcal{M}, w' \models^- A$ and $\mathcal{M}, w' \not\models^+ B$
cC_3	$\mathcal{M}, w \models^- (A \prec B)$	iff for every $w' \geq w : \mathcal{M}, w' \not\models^+ A$ or $\mathcal{M}, w' \models^+ B$
cC_4	$\mathcal{M}, w \models^- (A \prec B)$	iff for every $w' \geq w : \mathcal{M}, w' \models^- A$ or $\mathcal{M}, w' \not\models^- B$

Table: Support of falsity conditions for implications and co-implications

Observation (Persistence)

For every $\mathcal{L}$-formula A, model $\langle I, \leq, v^+, v^- \rangle$, and w, $w' \in I$:
if $w \leq w'$, then $w \models^+ A$ implies $w' \models^+ A$;
if $w \leq w'$, then $w \models^- A$ implies $w' \models^- A$.

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The different support of falsity conditions for implications and co-implications result in *sixteen* extensions of HB. Valid equivalences characteristic of these logics are stated in the next Table. The logics in the language $\mathcal{L}$ that differ from each other only with respect to validating a certain pair of these equivalences (one from the I-equivalences and one from the C-equivalences) are referred to as systems (I_i, C_j), $i, j \in \{1, 2, 3, 4\}$.

I_1	$\sim(A \to B)$	$\leftrightarrow$	$(A \wedge \sim B)$	neg. implication, classical reading
I_2	$\sim(A \to B)$	$\leftrightarrow$	$(A \to \sim B)$	neg. implication, connexive reading
I_3	$\sim(A \to B)$	$\leftrightarrow$	$(A \prec B)$	neg. implication as co-implication
I_4	$\sim(A \to B)$	$\leftrightarrow$	$(\sim B \prec \sim A)$	neg. implication as contraposed co-impl.
C_1	$\sim(A \prec B)$	$\leftrightarrow$	$(\sim A \vee B)$	neg. co-implication, classical reading
C_2	$\sim(A \prec B)$	$\leftrightarrow$	$(\sim A \prec B)$	neg. co-implication, connexive reading
C_3	$\sim(A \prec B)$	$\leftrightarrow$	$(A \to B)$	neg. co-implication as implication
C_4	$\sim(A \prec B)$	$\leftrightarrow$	$(\sim B \to \sim A)$	neg. co-implication as contraposed impl.

Table: Constructively negated implications and co-implications

Definition

The logics (I_i, C_j) are defined as the triples $(\mathcal{L}, \models^+_{I_i,C_j}, \models^-_{I_i,C_j})$, where the entailment relations $\models^+_{I_i,C_j}, \models^-_{I_i,C_j} \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{P}(\mathcal{L})$ are defined as follows:
$\Delta \models^+_{I_i,C_j} \Gamma$ iff for every model $\mathcal{M} = \langle I, \leq, v^+, v^- \rangle$ defined with clauses cI_i and cC_j and every $w \in I$, if $\mathcal{M}, w \models^+ A$ for every $A \in \Delta$, then $\mathcal{M}, w \models^+ B$ for some $B \in \Gamma$, and
$\Delta \models -_{I_i,C_j} \Gamma$ iff for every model $\mathcal{M} = \langle I, \leq, v^+, v^- \rangle$ defined with clauses cI_i and cC_j and every $w \in I$, if $\mathcal{M}, w \models^- A$ for every $A \in \Gamma$, then $\mathcal{M}, w \models^- B$ for some $B \in \Delta$.
For singleton sets $\{A\}$ and $\{B\}$, we write $A \models^+_{I_i,C_j} B$ $(A \models^-_{I_i,C_j} B)$ instead of $\{A\} \models^+_{I_i,C_j} \{B\}$ $(\{A\} \models^-_{I_i,C_j} \{B\})$. If the context is clear, we shall sometimes omit the subscript $_{I_i,C_j}$.

Observation

If $(I_i, C_j) \neq (I_4, C_4)$*, then* $\models^+_{I_i,C_j} \neq \models^-_{I_i,C_j}$.

Observation

If $(I_i, C_j) \neq (I_4, C_4)$, *then* $\models^{+}_{I_i,C_j} \neq \models^{-}_{I_i,C_j}$.

We do not require that for atomic formulas p, $v^{+}(p) \cap v^{-}(p) = \varnothing$. Therefore, the logics under consideration are *paraconsistent*. Neither is it the case that for any formula B, $\{p, \sim p\} \models^{+}_{I_i,C_j} B$ nor is it the case that $B \models^{-}_{I_i,C_j} \{p, \sim p\}$. (Co-negation is, of course, also a paraconsistent negation, whereas intuitionistic negation is 'paracomplete'.)

A formula is in *negation normal form* if it contains $\sim$ only in front of atoms. The following translations ρ_{I_i,C_j} send every formula A to a formula in negation normal form, where $p \in Atom$ and $\odot \in \{\vee, \wedge, \rightarrow, \prec\}$:

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$$
\begin{array}{lcl}
\rho_{I_i,C_j}(p) & = & p \\
\rho_{I_i,C_j}(\sim p) & = & \sim p \\
\rho_{I_i,C_j}(\sim\sim A) & = & \rho_{I_i,C_j}(A) \\
\rho_{I_i,C_j}(A \odot B) & = & \rho_{I_i,C_j}(A) \odot \rho_{I_i,C_j}(B) \\
\rho_{I_i,C_j}(\sim(A \vee B)) & = & \rho_{I_i,C_j}(\sim A) \wedge \rho_{I_i,C_j}(\sim B) \\
\rho_{I_i,C_j}(\sim(A \wedge B)) & = & \rho_{I_i,C_j}(\sim A) \vee \rho_{I_i,C_j}(\sim B) \\
\rho_{I_1,C_j}(\sim(A \rightarrow B)) & = & \rho_{I_1,C_j}(A) \wedge \rho_{I_1,C_j}(\sim B) \\
\rho_{I_2,C_j}(\sim(A \rightarrow B)) & = & \rho_{I_2,C_j}(A) \rightarrow \rho_{I_2,C_j}(\sim B) \\
\rho_{I_3,C_j}(\sim(A \rightarrow B)) & = & \rho_{I_3,C_j}(A) \prec \rho_{I_3,C_j}(B) \\
\rho_{I_4,C_j}(\sim(A \rightarrow B)) & = & \rho_{I_4,C_j}(\sim B) \prec \rho_{I_4,C_j}(\sim A) \\
\rho_{I_i,C_1}(\sim(A \prec B)) & = & \rho_{I_i,C_1}(\sim A) \vee \rho_{I_i,C_1}(B) \\
\rho_{I_i,C_2}(\sim(A \prec B)) & = & \rho_{I_i,C_2}(\sim A) \prec \rho_{I_i,C_2}(B) \\
\rho_{I_i,C_3}(\sim(A \prec B)) & = & \rho_{I_i,C_3}(A) \rightarrow \rho_{I_i,C_3}(B) \\
\rho_{I_i,C_4}(\sim(A \prec B)) & = & \rho_{I_i,C_4}(\sim B) \rightarrow \rho_{I_i,C_4}(\sim A)
\end{array}
$$

Lemma

For every formula A, $\rho_{I_i,C_j}(A)$ is in negation normal form and A $\models^+_{I_i,C_j} \rho_{I_i,C_j}(A)$, $\rho_{I_i,C_j}(A) \models^+_{I_i,C_j} A$, $A \models^-_{I_i,C_j} \rho_{I_i,C_j}(A)$, $\rho_{I_i,C_j}(A) \models^-_{I_i,C_j} A$.

We supplement the BHK interpretation by interpretations in terms of canonical disproofs, canonical reductions to absurdity (alias non-truth), and canonical reductions to non-falsity. That is, we define the notions of canonical proofs, disproofs, dual proofs and dual disproofs of complex $\mathcal{L}$-formulas by simultaneous induction.

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We will make the following assumptions:

- for no $\mathcal{L}$-formula A there exists both a proof and a dual proof of A;
- for no $\mathcal{L}$-formula A there exists both a disproof and a dual disproof of A;
- every $\mathcal{L}$-formula A either has a proof or dual proof;
- every $\mathcal{L}$-formula A either has a disproof or dual disproof.

- A canonical proof of a strongly negated formula $\sim A$ is a canonical disproof of A.
- A canonical proof of a conjunction $(A \wedge B)$ is a pair (π_1, π_2) consisting of a canonical proof π_1 of A and a canonical proof π_2 of B.
- A canonical proof of a disjunction $(A \vee B)$ is a pair (i, π) such that $i = 0$ and π is a canonical proof of A or $i = 1$ and π is a canonical proof of B.
- A canonical proof of an implication $(A \rightarrow B)$ is a construction that transforms any canonical proof of A into a canonical proof of B.
- A canonical proof of a co-implication $(A \prec B)$ is a pair (π_1, π_2), where π_1 is a canonical proof of A and π_2 is a canonical dual proof of B. (This pair is a canonical dual proof of $(A \rightarrow B)$.)

- A canonical disproof of a strongly negated formula $\sim A$ is a canonical proof of A.
- A canonical disproof of a conjunction $(A \wedge B)$ is a pair (i, π) such that $i = 0$ and π is a canonical disproof of A or $i = 1$ and π is a canonical disproof of B.
- A canonical disproof of a disjunction $(A \vee B)$ is a pair (π_1, π_2) consisting of a canonical disproof π_1 of A and a canonical disproof π_2 of B.

- A canonical disproof of an implication $(A \to B)$ in

$(I_1 C_j)$ is a pair (π_1, π_2) consisting of a canonical proof π_1 of A and a canonical disproof π_2 of B.

$(I_2 C_j)$ is a construction that transforms any canonical proof of A into a canonical disproof of B.

$(I_3 C_j)$ is a pair (π_1, π_2), where π_1 is a canonical proof of A and π_2 is a canonical dual proof of B. (This pair is a canonical dual proof of $(A \to B)$.)

$(I_4 C_j)$ is a pair (π_1, π_2), where π_1 is a canonical disproof of B and π_2 is a canonical dual disproof of A.

- A canonical disproof of a co-implication $(A \prec B)$ in

$(I_i C_1)$ is a pair (i, π) such that $i = 0$ and π is a canonical disproof of A or $i = 1$ and π is a canonical proof of B.

$(I_i C_2)$ is a pair (π_1, π_2), where π_1 is a canonical disproof of A and π_2 is a canonical dual proof of B. (This pair is a canonical dual proof of $(A \to \sim B)$.)

$(I_i C_3)$ is a construction that transforms any canonical proof of A into a canonical proof of B.

$(I_i C_4)$ is a construction that transforms any canonical disproof of B into a canonical disproof of A.

- A canonical reduction to non-truth (canonical dual proof) of a strongly negated formula $\sim A$ is canonical dual disproof of A.
- A canonical reduction to non-truth of a conjunction $(A \wedge B)$ is a pair (i, π) such that $i = 0$ and π is a canonical dual proof of A or $i = 1$ and π is a canonical dual proof of B.
- A canonical reduction to non-truth of a disjunction $(A \vee B)$ is a pair (π_1, π_2) consisting of a dual proof π_1 of A and a dual proof π_2 of B.
- A canonical reduction to non-truth of an implication $(A \rightarrow B)$ is a pair (π_1, π_2), where π_1 is a canonical proof of A and π_2 is a canonical dual proof of B. (This pair is a canonical proof of $(A \multimap\!\!\!< B)$.)
- A canonical reduction to non-truth of a co-implication $(A \multimap\!\!\!< B)$ is a construction that transforms any dual proof of B into a dual proof of A.

- A canonical reduction to non-falsity (canonical dual disproof) of a strongly negated formula $\sim A$ is a canonical dual proof of A.
- A canonical reduction to non-falsity of a conjunction $(A \wedge B)$ is a pair (π_1, π_2) consisting of a dual disproof π_1 of A and a dual disproof π_2 of B.
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- A canonical reduction to non-falsity of an implication $(A \rightarrow B)$ in

$(I_1 C_j)$ is a pair (i, π) such that $i = 0$ and π is a canonical dual proof of A or $i = 1$ and π is a canonical dual disproof of B.

$(I_2 C_j)$ is a pair (π_1, π_2), where π_1 is a canonical proof of A and π_2 is a canonical dual disproof of B.

$(I_3 C_j)$ is a pair (π_1, π_2), where π_1 is a canonical proof of A and π_2 is a canonical dual proof of B. (This pair is a canonical dual proof of $(A \rightarrow B)$.)

$(I_4 C_j)$ is a pair (π_1, π_2), where π_1 is a canonical disproof of B and π_2 is a canonical dual disproof of A. (This pair is a canonical dual proof of $(\sim B \rightarrow \sim A)$.)

- A canonical reduction to non-falsity of a co-implication $(A \prec B)$ in

$(I_i C_1)$ is a pair (π_1, π_2), where π_1 is a caninical dual disproof of A and π_2 is a canonical dual proof of B.

$(I_i C_2)$ is a construction that transforms any canonical dual proof of B into a canonical dual disproof of A. (This construction is a canonical dual proof $(\sim A \prec B)$.)

$(I_i C_3)$ is a pair (π_1, π_2), where π_1 is a canonical proof of A and π_2 is a canonical dual proof of B. (Thi spair is a canconcal dual proof of $(A \rightarrow B)$.)

$(I_i C_4)$ is a pair (π_1, π_2), where π_1 is a canonical disproof of B and π_2 is a canonical dual disproof of A.

To show by induction on the construction of inferences that the logics (I_i, C_j) are sound with respect to the above BHK-style interpretation in terms of proof, disproof, and their duals, we need proof systems for the semantically defined logics (I_i, C_j).

For example, we want to show that if $\sim A$ is provable, then there is a construction which is a disproof of A.

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For example, we want to show that if $\sim A$ is provable, then there is a construction which is a disproof of A.

We consider the the display calculi defined in (Wansing 2008).

The set of structures (or Gentzen terms) is defined as follows:

$$\begin{array}{rl} \text{formulas:} & A \in \mathit{Form(Atom)} \\ \text{structures} & X \in \mathit{Struc(Form)} \\ X ::= & A \mid \mathbf{I} \mid (X \circ X) \mid (X \bullet X). \end{array}$$

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The intended interpretation of the connective $\circ$ as conjunction in antecedent position and as implication in succedent position and of $\bullet$ as co-implication in antecedent position and as disjunction in succedent position justifies certain 'display postulates' (*dp*):

$$\frac{\dfrac{Y \vdash X \circ Z}{X \circ Y \vdash Z}}{X \vdash Y \circ Z} \qquad \frac{\dfrac{X \vdash Y \circ Z}{X \circ Y \vdash Z}}{Y \vdash X \circ Z} \qquad \frac{\dfrac{X \bullet Z \vdash Y}{X \vdash Y \bullet Z}}{X \bullet Y \vdash Z} \qquad \frac{\dfrac{X \bullet Y \vdash Z}{X \vdash Y \bullet Z}}{X \bullet Z \vdash Y}$$

Moreover, the interpretation of **I** as the empty structure suggests the following structural inference rules:

$$\frac{\dfrac{X \circ \mathbf{I} \vdash Y}{X \vdash Y}}{\mathbf{I} \circ X \vdash Y} \qquad \frac{\dfrac{\mathbf{I} \circ X \vdash Y}{X \vdash Y}}{X \circ \mathbf{I} \vdash Y} \qquad \frac{\dfrac{X \vdash Y \bullet \mathbf{I}}{X \vdash Y}}{X \vdash \mathbf{I} \bullet Y} \qquad \frac{\dfrac{X \vdash \mathbf{I} \bullet Y}{X \vdash Y}}{X \vdash Y \bullet \mathbf{I}}$$

In addition there are various 'logical' structural rules:

$$\frac{}{p \vdash p}\ (id)$$

$$\frac{}{\sim p \vdash \sim p}\ (id\sim)$$

$$\frac{X \vdash A \quad A \vdash Y}{X \vdash Y}\ (cut)$$

and versions of the familiar structural rules from standard Gentzen systems for classical logic, monotonicity, exchange, and contraction, plus associativity:

$$\frac{X \vdash Y}{X \vdash Y \bullet Z}\ (rm) \qquad \frac{X \vdash Y}{X \circ Z \vdash Y}\ (lm)$$

$$\frac{X \vdash Y \bullet Z}{X \vdash Z \bullet Y}\ (re) \qquad \frac{X \circ Z \vdash Y}{Z \circ X \vdash Y}\ (le)$$

$$\frac{X \vdash Y \bullet Y}{X \vdash Y}\ (rc) \qquad \frac{X \circ X \vdash Y}{X \vdash Y}\ (lc)$$

$$\frac{X \vdash (Y \bullet Z) \bullet X'}{X \vdash Y \bullet (Z \bullet X')}\ (ra) \qquad \frac{(X \circ Y) \circ Z \vdash X'}{X \circ (Y \circ Z) \vdash X'}\ (la)$$

Table: Structural sequent rules

$$\frac{X \vdash A \quad Y \vdash B}{X \circ Y \vdash (A \wedge B)}\ (\vdash \wedge) \qquad \frac{A \circ B \vdash X}{(A \wedge B) \vdash X}\ (\wedge \vdash)$$

$$\frac{X \vdash A \bullet B}{X \vdash (A \vee B)}\ (\vdash \vee) \qquad \frac{A \vdash X \quad B \vdash Y}{(A \vee B) \vdash X \bullet Y}\ (\vee \vdash)$$

$$\frac{X \vdash A \circ B}{X \vdash (A \to B)}\ (\vdash \to) \qquad \frac{X \vdash A \quad B \vdash Y}{(A \to B) \vdash X \circ Y}\ (\to \vdash)$$

$$\frac{X \vdash B \quad A \vdash Y}{X \bullet Y \vdash B \prec A}\ (\vdash \prec) \qquad \frac{B \bullet A \vdash X}{B \prec A \vdash X}\ (\prec \vdash)$$

$$\frac{X \vdash \sim A \bullet \sim B}{X \vdash \sim(A \wedge B)}\ (\vdash \sim\wedge) \qquad \frac{\sim A \vdash X \quad \sim B \vdash Y}{\sim(A \wedge B) \vdash X \bullet Y}\ (\sim\wedge \vdash)$$

$$\frac{X \vdash \sim A \quad Y \vdash \sim B}{X \circ Y \vdash \sim(A \vee B)}\ (\vdash \sim\vee) \qquad \frac{\sim A \circ \sim B \vdash X}{\sim(A \vee B) \vdash X}\ (\sim\vee \vdash)$$

$$\frac{X \vdash A}{X \vdash \sim\sim A}\ (\vdash \sim\sim) \qquad \frac{A \vdash X}{\sim\sim A \vdash X}\ (\sim\sim \vdash)$$

Table: Introduction rules shared by all logics (I_i, C_j)

rI_1	$\dfrac{X \vdash A \quad Y \vdash \sim B}{X \circ Y \vdash \sim(A \rightarrow B)}$	$\dfrac{A \circ \sim B \vdash X}{\sim(A \rightarrow B) \vdash X}$
rI_2	$\dfrac{X \vdash A \circ \sim B}{X \vdash \sim(A \rightarrow B)}$	$\dfrac{X \vdash A \quad \sim B \vdash Y}{\sim(A \rightarrow B) \vdash X \circ Y}$
rI_3	$\dfrac{X \vdash A \quad B \vdash Y}{X \bullet Y \vdash \sim(A \rightarrow B)}$	$\dfrac{A \bullet B \vdash X}{\sim(A \rightarrow B) \vdash X}$
rI_4	$\dfrac{X \vdash \sim B \quad \sim A \vdash Y}{X \bullet Y \vdash \sim(A \rightarrow B)}$	$\dfrac{\sim B \bullet \sim A \vdash X}{\sim(A \rightarrow B) \vdash X}$
rC_1	$\dfrac{X \vdash \sim A \bullet B}{X \vdash \sim(A \prec B)}$	$\dfrac{\sim A \vdash X \quad B \vdash Y}{\sim(A \prec B) \vdash X \bullet Y}$
rC_2	$\dfrac{X \vdash \sim A \quad B \vdash Y}{X \bullet Y \vdash \sim(A \prec B)}$	$\dfrac{\sim A \bullet B \vdash X}{\sim(A \prec B) \vdash X}$
rC_3	$\dfrac{X \vdash A \circ B}{X \vdash \sim(A \prec B)}$	$\dfrac{Y \vdash A \quad B \vdash X}{\sim(A \prec B) \vdash Y \circ X}$
rC_4	$\dfrac{X \vdash \sim B \circ \sim A}{X \vdash \sim(A \prec B)}$	$\dfrac{Y \vdash \sim B \quad \sim A \vdash X}{\sim(A \prec B) \vdash Y \circ X}$

Table: Sequent rules for negated implications and co-implications

The display sequent calculi $\delta(I_i, C_j)$, $i, j \in \{1, 2, 3, 4\}$, for the constructive logics (I_i, C_j) share the display postualtes, the structural rules and the introduction rules stated in the penultimate table. The particular display calculus $\delta(I_i, C_j)$ then is the proof system obtained by adding the rules rI_i and rC_j from the preceding table.

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A derivation of a sequent s from a set of sequents $\{s_1, \ldots, s_n\}$ in $\delta(I_i, C_j)$ is defined as a tree with root s such that every leaf is an instantiation of (id), $(id{\sim})$, or a sequent from $\{s_1, \ldots, s_n\}$, and every other node is obtained by an application of one of the remaining rules. A proof of a sequent s in $\delta(I_i, C_j)$ is a derivation of s from $\varnothing$. Sequents s and s′ are said to be interderivable iff s is derivable from $\{s'\}$ and s′ is derivable from s.

Two sequents s and s′ are said to be structurally equivalent if they are interderivable by means of display postulates only. It is characteristic for display calculi that any substructure of a given sequent s may be displayed as the entire antecedent or succedent of a structurally equivalent sequent s′.

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If $s = X \vdash Y$ is a sequent, then the displayed occurrence of X (Y) is an antecedent (succedent) part of s. If an occurrence of $(Z \circ W)$ is an antecedent part of s, then the displayed occurrences of Z and W are antecedent parts of s. If an occurrence of $(Z \bullet W)$ is an antecedent part of s, then the displayed occurrence of Z (W) is an antecedent (succedent) part of s. If an occurrence of $(Z \circ W)$ is a succedent part of s, then the displayed occurrence of Z (W) is an antecedent (succedent) part of s. If an occurrence of $(Z \bullet W)$ is a succedent part of s, then the displayed occurrences of Z and W are succedent parts of s.

Theorem

For every sequent s *and every antecedent (succedent) part* X *of* s*, there exists a sequent* s′ *structurally equivalent to* s *such that* X *is the entire antecedent (succedent) of* s′*.*

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Observation

For every $\mathcal{L}$*-formula* A *and every calculus* $\delta(I_i, C_j)$*,* $A \vdash A$ *is provable.*

One can define translations τ_1 and τ_2 from structures into formulas such that these translations reflect the intuitive, context-sensitive interpretation of the structural connectives: τ_1 translates structures which are antecedent parts of a sequent, whereas τ_2 translates structures which are succedent parts of a sequent.

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Definition

The translations τ_1 and τ_2 from structures into formulas are inductively defined as follows, where A is a formula and p is a certain atom:

$$
\begin{array}{rclrcl}
\tau_1(A) & = & A & \tau_2(A) & = & A \\
\tau_1(\mathbf{I}) & = & p \rightarrow p & \tau_2(\mathbf{I}) & = & p \prec p \\
\tau_1(X \circ Y) & = & \tau_1(X) \wedge \tau_1(Y) & \tau_2(X \circ Y) & = & \tau_1(X) \rightarrow \tau_2(Y) \\
\tau_1(X \bullet Y) & = & \tau_1(X) \prec \tau_2(Y) & \tau_2(X \bullet Y) & = & \tau_2(X) \vee \tau_2(Y)
\end{array}
$$

Theorem (Soundness)

(1) If $X \vdash Y$ *is provable in* $\delta(I_i, C_j)$, *then* $\tau_1(X) \models^{+}_{I_i,C_j} \tau_2(Y)$.
(2) If $X \vdash Y$ *is provable in* $\delta(I_i, C_j)$, *then* $\sim\tau_2(Y) \models^{-}_{I_i,C_j} \sim\tau_1(X)$.

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The language $\mathcal{L}^*$ results from $\mathcal{L}$ by adding for every atomic formula p a new atom p^*. If A is an $\mathcal{L}$-formula, $(A)^*$ is the result of replacing every strongly negated atom $\sim p$ in A by p^*.

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Lemma

For every $\mathcal{L}$-formula A, if $\varnothing \models^{+}_{I_i,C_j} A$, then $(\rho_{I_i,C_j}(A))^*$ is valid in HB.

Lemma

For every $\sim$-free $\mathcal{L}$-formula A, if A is provable in HB*, then* $\mathbf{I} \vdash A$ *is provable in* $\delta(I_i, C_j)$ *without using any sequent rules for strongly negated formulas.*

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For every $\sim$-free $\mathcal{L}$-formula A, if A is provable in HB*, then* $\mathbf{I} \vdash A$ *is provable in* $\delta(I_i, C_j)$ *without using any sequent rules for strongly negated formulas.*

Lemma

For every $\mathcal{L}$-formula A, $A \vdash \rho_{I_i,C_j}(A)$ *and* $\rho_{I_i,C_j}(A) \vdash A$ *are provable in* $\delta(I_i, C_j)$.

Lemma

For every $\sim$*-free* $\mathcal{L}$*-formula* A*, if* A *is provable in* HB*, then* $\mathbf{I} \vdash A$ *is provable in* $\delta(I_i, C_j)$ *without using any sequent rules for strongly negated formulas.*

Lemma

For every $\mathcal{L}$*-formula* A*,* $A \vdash \rho_{I_i,C_j}(A)$ *and* $\rho_{I_i,C_j}(A) \vdash A$ *are provable in* $\delta(I_i, C_j)$*.*

Lemma

Every sequent $X \vdash \tau_1(X)$ *and* $\tau_2(X) \vdash X$ *is provable in* $\delta(I_i, C_j)$*, for all* $i, j \in \{1, 2, 3, 4\}$*.*

Theorem (Completeness)

(1) If $\rho_{I_i,C_j}(\tau_1(X)) \models^{+}_{I_i,C_j} \rho_{I_i,C_j}(\tau_2(Y))$*, then* $X \vdash Y$ *is provable in* $\delta(I_i, C_j)$*. (2) If* $\rho_{I_i,C_j}(\sim\tau_2(Y)) \models^{-}_{I_i,C_j} \rho_{I_i,C_j}(\sim\tau_1(X))$*, then* $X \vdash Y$ *is provable in* $\delta(I_i, C_j)$*.*

Theorem (Completeness)

(1) If $\rho_{I_i,C_j}(\tau_1(X)) \models^{+}_{I_i,C_j} \rho_{I_i,C_j}(\tau_2(Y))$, then $X \vdash Y$ is provable in $\delta(I_i, C_j)$. (2) If $\rho_{I_i,C_j}(\sim\tau_2(Y)) \models^{-}_{I_i,C_j} \rho_{I_i,C_j}(\sim\tau_1(X))$, then $X \vdash Y$ is provable in $\delta(I_i, C_j)$.

Let $\delta(I_i, C_j)^+$ denote the result of dropping all sequent rules exhibiting $\sim$ from $\delta(I_i, C_j)$.

Theorem

If $X \vdash Y$ is provable in system $\delta(I_i, C_j)$, then $(\rho_{Ii,C_j}(\tau_1(X)))^* \vdash (\rho_{Ii,C_j}(\tau_2(Y)))^*$ is provable in $\delta(I_i, C_j)^+$ without any applications of (cut).

Theorem

Let $i, j \in \{1, 2, 3, 4\}$. If $X \vdash Y$ is provable in $\delta(I_i, C_j)$, then

1. there exists a construction π such that $\pi(\pi')$ is a canonical proof of $\tau_2(Y)$ whenever π' is a canonical proof of $\tau_1(X)$.

2. there exists a construction π such that $\pi(\pi')$ is a canonical dual proof of $\tau_1(X)$ whenever π' is a canonical dual proof of $\tau_2(Y)$.

Theorem

Let $i, j \in \{1, 2, 3, 4\}$.

- *If $\mathbf{I} \vdash A$ is provable in $\delta(I_i, C_j)$, then there exists a construction π which is a proof of A.*
- *If $A \vdash \mathbf{I}$ is provable in $\delta(I_i, C_j)$, then there exists a construction π which is a dual proof of A.*
- *If $\mathbf{I} \vdash {\sim}A$ is provable in $\delta(I_i, C_j)$, then there exists a construction π which is a disproof of A.*
- *If ${\sim}A \vdash \mathbf{I}$ is provable in $\delta(I_i, C_j)$, then there exists a construction π which is a dual disproof of A.*

Theorem

Let $i, j \in \{1, 2, 3, 4\}$.

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Proof. Any canonical proof of $\tau_1(\mathbf{I}) = (p \to p)$ and any canonical dual proof of $\tau_2(\mathbf{I}) = (p \prec p)$ is the identity function. Every disproof of A is a proof of ${\sim}A$ and every canonical dual disproof of A is a canonical dual proof of ${\sim}A$. q.e.d.

(propositional) logic	*soundness with respect to an interpretation*
intuitionistic logic	in terms of proofs
Nelson's logics	in terms of proofs and disproofs
dual intuitionistic logic	in terms of dual proofs
bi-intuitionistic logic	in terms of proofs and dual proofs
bi-intuitionistic logic extended by strong negation	in terms of proof, disproofs, and their duals

Table: Summary